A TREATISE

ON

THE INTEGRAL CALCULUS.

PART I.

CONTAINING AN

ELEMENTARY ACCOUNT OF ELLIPTIC INTEGRALS AND APPLICATIONS TO PLANE CURVES;

With numerous Examples.

BY

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INTEGRAL CALCULUS.

CHAPTER I.

ELEMENTARY INTEGRALS.

1. The idea of the Integral Calculus arises naturally out of that of the Differential, so that, at least in the more elementary parts of the subject, we require no new investigation of principles, but depend entirely for them on the Differential Calculus. The principal object of the Integral Calculus is to find the value of a function of a single variable when its differential coefficient is given, and this is what we especially propose to consider in this Treatise. From a more general point of view the object is to discover the relations which exist between several variables and functions of these variables, from given equations connecting the variables, the functions, and the successive differential coefficients of the functions. Such equations occur in various problems in Physics and Geometry, their solution forming a special department in itself under the name of the Theory of Differential Equations. In the case of a single variable, these equations are considered as solved when they are made to depend upon others of the form just referred to, namely, $\frac{dy}{dx} = f(x)$, where $f(x)$ is the given expression which is

supposed to be the differential coefficient of the unknown function y. The required function y is called the integral of $f(x)\,dx$, and is expressed by the notation $y = \int f(x)\,dx$. The expression $f(x)\,dx$, which is called an element of the integral, is the limiting value of the increment of y when we substitute $x + dx$ for x. The integral y may thus be regarded as the sum of an infinite number of these elements. It was from this point of view that the word integral and the symbol $\int$ arose, an integral being the total or integral sum of an infinite number of elements, and the symbol $\int$ being the initial letter of the word sum, in the same way as the symbol of differentiation is the initial letter of the word difference. The simplest method, however, of finding an integral, or of integrating a given differential, is by regarding the process as the inverse of differentiation, and is that which we shall almost exclusively make use of in this Treatise.

2. Since any arbitrary quantity C which does not vary with x disappears in differentiation, we must add on to y such an arbitrary constant, in order to find the general value of the integral of a given differential $f(x)\,dx$. In finding integrals we shall usually omit for convenience the constant C, but it must always be considered as involved in each case. In all the applications of the Integral Calculus the constant is of great importance, and its value must be determined in each case by the conditions of the problem.

Since $\frac{d}{dx}(ay) = a\frac{dy}{dx}$, where a is a constant quantity, we have $\int\left(a\frac{dy}{dx}\,dx\right) = ay$, from which it follows, that if a constant multiply a differential coefficient as a factor, it will also multiply its integral. Again, let y_1, y_2 be functions of x,

and u_1, u_2 their differential coefficients, so that $\frac{dy_1}{dx} = u_1$, and $y_1 = \int u_1 dx$, &c.; then

$$\frac{d}{dx}(y_1 \pm y_2) = u_1 \pm u_2;$$

hence

$$\int (u_1 \pm u_2)\, dx = y_1 \pm y_2 = \int u_1 dx \pm \int u_2 dx,$$

that is, the integral of the sum or difference of two differentials is equal to the sum or difference of the integrals of these differentials.

3. The most important class of integrals are those of algebraic functions of the variable. These can be expressed by algebraic expressions, by logarithms, or by angles determined by their circular measures, besides an infinite number of other functions which can be defined in no other manner than as the integrals of given algebraic expressions. Amongst the latter are included the functions called elliptic, hyperelliptic, and Abelian. It may be remarked that, without making any use of the Integral Calculus, we can arrive at the ideas of circular functions and logarithms from trigonometrical and algebraic considerations; but we might consider these functions altogether as derived from the integrals of certain algebraic expressions, and thence deduce their properties, as we do in the case of the elliptic functions and the higher transcendents.

Our main object is then to reduce given integrals to those irreducible fundamental forms, and, in fact, the greater part of this and a few of the following chapters consists of an enumeration of those cases in which it is possible to effect the reduction to the elementary algebraic, circular, and logarithmic forms, and an explanation of the means employed for that purpose.

4. In a great number of practical applications of the Integral Calculus it becomes necessary to determine the actual numerical values of the integrals when certain values of the variable are assigned. In the case of algebraic functions these values are of course known at once, but in the case of circular functions, logarithms, and elliptic functions, we must have recourse to the proper tables.

The integrals of expressions involving circular, logarithmic, or exponential functions will frequently be capable of expression in terms of similar functions, but if not, can be made to depend upon certain irreducible transcendents. It may be observed that circular functions are often introduced into algebraic expressions in order to give them a simpler form, so that many differentials, though apparently involving these functions, really depend upon algebraic quantities.

When a proposed integral cannot be obtained in a finite form in terms of algebraic quantities, or those functions whose values are tabulated, in order to obtain its numerical value, we must expand the differential expression in an infinite converging series, and then integrate each term separately.

5. We may now enumerate the different methods by which the reduction of integrals to the elementary forms is effected. These are—

(1) Transformation by the introduction of a new variable, that is, supposing we have an integral $\int f(x)\,dx$, then putting $x = \phi(z)$, we get $dx = \phi'(z)\,dz$ and $\int f(x)\,dx = \int f\{\phi(z)\}\,\phi'(z)\,dz$, which, by a proper assumption of the function $\phi(z)$, is reduced to one of the elementary forms.

(2) Integration by the resolution of rational expressions into the sums of several others of a similar form.

(3) Rationalization, that is, the substitution of a new variable in irrational expressions, so as to make them rational. This is a particular case of (1).

(4). The use of formulae of reduction, by which means a proposed integral is made to depend upon one more simple, and this again in the same way on another yet more simple, and so on, until finally we arrive at one of the elementary forms.

(5) Integration by parts, that is, the use of a certain formula obtained thus:—By the Differential Calculus we have

$$d(uv) = udv + vdu,$$

whence we get

$$\int udv = uv - \int vdu. \qquad (1)$$

We thus see that the formula of integration by parts makes a given integral $\int udv$ depend upon another, namely, $\int vdu$.

6. From the simplest considerations of the Differential Calculus we can write down at once the following elementary integrals:—

$$\int x^m dx = \frac{x^{m+1}}{m+1}, \qquad (2)$$

where m is any constant quantity whatever, except negative unity.

$$\int \frac{dx}{x} = \log(x), \qquad (3)$$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}, \qquad (4)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}. \qquad (5)$$

Either of the latter pair of integrals may be taken as the fundamental elementary form for circular functions; but it is evident that they cannot be independent, that is, one must be capable of being transformed into the other by an algebraic substitution. In fact, putting $x = a \sin \theta$, and taking another variable z, so that $z = a \tan \frac{1}{2}\theta$, we have, by differentiation,

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = d\theta, \quad \frac{dz}{a^2 + z^2} = \frac{d\theta}{2a},$$

whence
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = 2a \int \frac{dz}{a^2 + z^2}.$$

Expressing then x in terms of z, we see that the substitution $x = 2a^2 z / (a^2 + z^2)$ transforms (4) to the form (5). This transformation is of importance, as it enables us to rationalize an expression involving the radical $\sqrt{(a^2 - x^2)}$, thus affording an example of the method (3) of the preceding Article.

Examples.

1. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}.$

2. $\int \frac{dx}{x^2} = -\frac{1}{x}.$

3. $\int (a + bx)^n dx = \frac{1}{b} \int (a + bx)^n d(a + bx) = \frac{(a + bx)^{n+1}}{b(n + 1)}.$

4. $\int \frac{dx}{a + bx} = \frac{1}{b} \log (a + bx).$

5. $\int \frac{x\,dx}{\sqrt{(a^2 + x^2)}} = \sqrt{(a^2 + x^2)}.$

6. $\int \frac{(a - x)\,dx}{\sqrt{(2ax - x^2)}} = \frac{1}{2} \int \frac{d(2ax - x^2)}{\sqrt{(2ax - x^2)}} = \sqrt{(2ax - x^2)}.$

7. $\int (a+bx^n)^m x^{n-1}dx = \frac{1}{nb}\int (a+bx^n)^m\, d(a+bx^n) = \frac{(a+bx^n)^{m+1}}{nb(m+1)}.$

8. $\int \frac{x^{n-1}dx}{a+bx^n} = \frac{1}{nb}\log(a+bx^n).$

9. $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{dx}{x^3}\frac{1}{(1+a^2x^{-2})^{\frac{3}{2}}} = -\frac{1}{2a^2}\int \frac{d(1+a^2x^{-2})}{(1+a^2x^{-2})^{\frac{3}{2}}} = \frac{1}{a^2}(1+a^2x^{-2})^{-\frac{1}{2}}$

$$= \frac{x}{a^2\sqrt{(a^2+x^2)}}.$$

10. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{(a^2-x^2)}}.$

11. $\int \frac{dx}{(1+x^n)^{\frac{n+1}{n}}} = \frac{x}{(1+x^n)^{\frac{1}{n}}}.$

12. $\int -\frac{dx}{\sqrt{(a^2-x^2)}} = \cos^{-1}\frac{x}{a}.$

13. $\int -\frac{dx}{a^2+x^2} = \frac{1}{a}\cot^{-1}\frac{x}{a}.$

7. By means of the Differential Calculus we obtain the following elementary integrals in the case of circular and exponential functions:—

$$\int \sin x\,dx = -\cos x, \quad \int \cos x\,dx = \sin x. \tag{6}$$

$$\int \frac{dx}{\cos^2 x} = \tan x, \qquad \int \frac{dx}{\sin^2 x} = -\cot x. \tag{7}$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax}. \tag{8}$$

It may be observed, however, that none of these integrals are really fundamental, as they are all immediately reducible to the integrals of algebraic differentials; for instance, putting $\sin x = z$, the equation $\int \cos x\,dx = \sin x$ becomes merely $\int dz = z$.

EXAMPLES.

1. $\int \cos mx\,dx = \frac{1}{m}\sin mx.$

2. $\int \sin mx\,dx = \frac{1}{m}\cos mx.$

3. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta} = \sec\theta.$

4. $\int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\operatorname{cosec}\theta.$

5. $\int \tan\theta\,d\theta = \log\sec\theta.$

6. $\int \cot\theta\,d\theta = \log\sin\theta.$

7. $\int \tan^2\theta\,d\theta = (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta.$

8. $\int \cot^2\theta\,d\theta = -(\theta + \cot\theta).$

9. $\int \sec^4\theta\,d\theta = \int (1 + \tan^2\theta)\,d\tan\theta = \tan\theta + \frac{1}{3}\tan^3\theta.$

10. $\int \cos^2\theta\,d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta).$

11. $\int \frac{d\theta}{\sin^2\theta\cos^2\theta} = \tan\theta - \cot\theta.$

12. $\int \frac{\sin\theta\,d\theta}{a - b\cos\theta} = \frac{1}{b}\log(a - b\cos\theta).$

13. $\int \frac{dx}{\sqrt{(e^{2x} - 1)}} = \cos^{-1}(e^{-x}).$

8. Returning to algebraic functions, we observe that we can find the value of $\int y\,dx$, where y is a polynomial expression of the form

$$ax^m + bx^n + cx^p + \&c.,$$

where m, n, p are any quantities whatever.

Integrating each term separately, we obtain

$$\int y\,dx = \frac{ax^{m+1}}{m+1} + \frac{bx^{n+1}}{n+1} + \frac{cx^{p+1}}{p+1} + \&c., \tag{9}$$

in which all the terms are algebraic, unless y involves a term of the form gx^{-1}, the integral of which is $g \log x$.

To the preceding case many integrals can be reduced by the substitution of a new variable. For instance, to obtain the integral

$$\int \frac{x^m\,dx}{(a+bx)^n},$$

where m is a positive integer, we put $a + bx = z$, when it becomes

$$\frac{1}{b^{m+1}} \int (z-a)^m \frac{dz}{z^n}.$$

Expanding then $(z - a)^m$ by the binomial theorem, and integrating each term separately, the required integral is obtained.

Again, in the same way, expressions of the form

$$(a + bx^n)^{\frac{p}{q}} x^{m-1}\,dx$$

can be integrated immediately if m/n is a positive integer, or $m/n + p/q$ a negative integer. Putting $a + bx^n = z$, this expression becomes

$$\frac{1}{nb^{\frac{m}{n}}} z^{\frac{p}{q}} (z - a)^{\frac{m}{n}-1} dz,$$

which is immediately integrable by the expansion of $(z - a)^{\frac{m}{n}-1}$ in a limited number of terms.

Also $$(a+bx^n)^{\frac{p}{q}}x^{m-1}\,dx = (ax^{-n}+b)^{\frac{p}{q}}x^{m-1+\frac{np}{q}}\,dx$$

$$= -\frac{1}{n}a^{\left(\frac{m}{n}+\frac{p}{q}\right)}z^{\frac{p}{q}}(z-b)^{-\left(1+\frac{m}{n}+\frac{p}{q}\right)}dz, \text{ where } b+ax^{-n}=z.$$

Hence, as we have stated above, if $m/n + p/q$ is a negative integer, the expression to be integrated admits of expansion in a finite number of terms, each of which can be integrated at once.

Examples.

1. $\displaystyle\int \frac{dx}{x(x+a)} = \frac{1}{a}\log\left(\frac{x}{x+a}\right).$

2. $\displaystyle\int x\sqrt{(a-x)}\,dx = \frac{2}{5}(a-x)^{\frac{5}{2}} - \frac{2a}{3}(a-x)^{\frac{3}{2}}.$

3. $\displaystyle\int \frac{x\,dx}{(a+bx)^{\frac{2}{3}}} = \frac{3}{4b^2}(bx-3a)(a+bx)^{\frac{1}{3}}.$

4. $\displaystyle\int \frac{x\,dx}{(a+bx)^3} = -\frac{1}{2b^2}\frac{(2bx+a)}{(a+bx)^2}.$

5. $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^{\frac{1}{3}}} = -\frac{3}{20}(3a^2+2x^2)(a^2-x^2)^{\frac{2}{3}}.$

6. $\displaystyle\int \frac{dx}{x^m(a+bx^n)} = \frac{1}{a^{m+n-2}}\int \frac{(b+ax^{-1}-b)^{m+n-2}}{(b+ax^{-1})^n}x^{-2}\,dx$

$$= -\frac{1}{a^{m+n-1}}\int (b+ax^{-1}-b)^{m+n-2}\frac{d(b+ax^{-1})}{(b+ax^{-1})^n}.$$

7. $\displaystyle\int \frac{dx}{x^2(a+bx)^2} = \frac{1}{a^3}\left\{2\log\left(b+\frac{a}{x}\right) - \frac{a(a+2bx)}{x(a+bx)}\right\}.$

8. $\displaystyle\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left\{\frac{b^2x^2}{2(a+bx)^2} - \frac{2bx}{a+bx} - \log\left(b+\frac{a}{x}\right)\right\}.$

9. $\displaystyle\int \frac{dx}{x^4\sqrt{(1+x^2)}} = \left(\frac{2x^2-1}{3x^3}\right)\sqrt{(1+x^2)}.$

10. $\displaystyle\int \frac{dx}{x^2\sqrt{(1-x^2)}} = -\frac{1}{x}\sqrt{(1-x^2)}.$

9. To find the integral of $\frac{dx}{x^2 - a^2}$.

We have $$\frac{2a}{x^2 - a^2} = \frac{1}{x - a} - \frac{1}{x + a};$$
hence
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left\{\log(x - a) - \log(x + a)\right\} = \frac{1}{2a}\log\left(\frac{x - a}{x + a}\right). \quad (10)$$

In the same way we find
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left(\frac{a + x}{a - x}\right). \quad (11)$$

In practical applications the distinction between these two integrals becomes important, as in (10) x is supposed to be always greater than a, and in (11) less. Adding the two expressions together, we see that the sum of the two constants involved in each case is the imaginary quantity
$$-\frac{1}{2a}\log(-1).$$

In the same way, since
$$\frac{1}{(x - a)(x - b)} = \frac{1}{a - b}\left\{\frac{1}{x - a} - \frac{1}{x - b}\right\},$$
we find
$$\int \frac{dx}{(x - a)(x - b)} = \frac{1}{a - b}\log\left(\frac{x - a}{x - b}\right), \text{ or } \frac{1}{a - b}\log\left(\frac{a - x}{x - b}\right),$$

which might, however, be at once reduced to the preceding cases by putting $x = \frac{1}{2}(a + b) + z$.

It may be observed that the foregoing integrals afford simple examples of the application of method (2) of Art. 5.

10. We may now integrate the general form

$$\frac{dx}{a + 2bx + cx^2}.$$

We may write this

$$\frac{c\,dx}{(cx + b)^2 + ac - b^2}, \text{ which becomes } \frac{dz}{z^2 + ac - b^2},$$

if we substitute z for $cx + b$.

Hence, the expression to be integrated is reduced to the form (5) or (10) according as $ac - b^2$ is positive or negative. We find thus, if $ac - b^2$ is positive,

$$\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{\sqrt{(ac - b^2)}} \tan^{-1} \frac{cx + b}{\sqrt{(ac - b^2)}}; \tag{12}$$

and if $ac - b^2$ is negative,

$$\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{2\sqrt{(b^2 - ac)}} \log \left\{\frac{cx + b - \sqrt{(b^2 - ac)}}{cx + b + \sqrt{(b^2 - ac)}}\right\}. \tag{13}$$

EXAMPLES.

1. $\int \frac{dx}{(x-1)(x-2)} = \log\left(\frac{x-2}{x-1}\right).$

2. $\int \frac{dx}{(x+2)(x-4)} = \frac{1}{6}\log\left(\frac{x-4}{x+2}\right).$

3. $\int \frac{dx}{1 - x + x^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$

4. $\int \frac{(l + mx)\,dx}{a + 2bx + cx^2} = \frac{m}{c}\int \frac{(b + cx)\,dx}{a + 2bx + cx^2} + \frac{(lc - mb)}{c}\int \frac{dx}{a + 2bx + cx^2}$

$$= \frac{m}{2c}\log(a + 2bx + cx^2) + \frac{(lc - mb)}{c}\int \frac{dx}{a + 2bx + cx^2}.$$

5. $\int \frac{(l + mx)\,dx}{x^2 - 2ax + a^2 + \beta^2} = \frac{l + ma}{\beta}\tan^{-1}\left(\frac{x - a}{\beta}\right) + \frac{1}{2}m\log\{(x - a)^2 + \beta^2\}.$

6. $\displaystyle\int \frac{dx}{x^2 - 2x\cos\theta + 1} = \frac{1}{\sin\theta}\tan^{-1}\left(\frac{x - \cos\theta}{\sin\theta}\right).$

7. $\displaystyle\int \frac{x\,dx}{x^2 - 2x\cos\theta + 1} = \log\sqrt{(x^2 - 2x\cos\theta + 1)} + \cot\theta\tan^{-1}\left(\frac{x - \cos\theta}{\sin\theta}\right).$

8. $\displaystyle\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{\sqrt{(ac - b^2)}}\cos^{-1}\left\{\frac{ac - b^2}{c(a + 2bx + cx^2)}\right\}^{\frac{1}{2}}.$

9. $\displaystyle\int \frac{(l + mx)\,dx}{a + bx^2} = \frac{l}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right) + \frac{m}{b}\log(a + bx^2)^{\frac{1}{2}}.$

11. It may be noticed here that the integral of dx/x can be deduced from that of $x^m dx$. If we suppose the integral of the latter expression to vanish when $x = a$, we have

$$\int x^m dx = \frac{x^{m+1} - a^{m+1}}{m + 1}.$$

If we expand now x^{m+1} and a^{m+1} by the exponential series, that is, putting

$$x^h = 1 + h\log x + \frac{h^2}{\underline{|2}}(\log x)^2 + \&c.,$$

and a similar value for a^h, where $m + 1 = h$, we get

$$\int x^{h-1} dx = \frac{1}{h}(x^h - a^h) = \log x - \log a + \frac{h}{\underline{|2}}\{(\log x)^2 - (\log a)^2\} + \&c.$$

Hence, letting $h = 0$, we have, finally,

$$\int \frac{dx}{x} = \log x - \log a = \log\left(\frac{x}{a}\right), \text{ or } \log x,$$

if a is taken equal to unity.

12. Again, we can make circular functions depend upon logarithmic by means of an imaginary transformation.

In (11), putting $a = 1$, and $x = iz$, where $i^2 = -1$, we get

$$\int \frac{dz}{1+z^2} = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right),$$

and the constant to be added is zero if the integral vanishes with z. Hence, from (5) we have

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right),$$

that is, the circular function $\tan^{-1} z$ is expressed as a logarithm by the use of the imaginary i.

If we put $z = \tan\theta$ in the foregoing equation, we arrive at the well-known exponential values for $\sin\theta$ and $\cos\theta$.

13. As has been remarked already, we can define logarithmic and circular functions by means of the Integral Calculus solely, and thence derive all their properties. Thus, writing $\int dx/x = f(x)$, and if the constant is supposed to be such that $f(1)$ vanishes, we have

$$f(x) + f(y) = \int\left(\frac{dx}{x} + \frac{dy}{y}\right) = \int \frac{x\,dy + y\,dx}{xy} = \int \frac{d(xy)}{xy};$$

therefore

$$f(x) + f(y) = f(xy),$$

which may be considered as the fundamental property of the logarithmic function. We can then immediately derive the properties of $\tan^{-1} x$ by means of the expression of this function as a logarithm. Or, directly thus for $\sin^{-1} x$; putting

$$\int \frac{dx}{\sqrt{(1-x^2)}} = f(x),$$

and supposing the integral to vanish with x, if $f(x)+f(y) = f(z)$, a constant, we have $d\{f(x)+f(y)\}=0$, that is

$$\frac{dx}{\sqrt{(1-x^2)}}+\frac{dy}{\sqrt{(1-y^2)}}=0, \text{ or } \sqrt{(1-y^2)}\,dx+\sqrt{(1-x^2)}\,dy=0.$$

Now, integrating the terms of the latter expression by parts, namely, making use of the formula (1), we have

$$\int\sqrt{(1-y^2)}\,dx = x\sqrt{(1-y^2)}+\int\frac{xy\,dy}{\sqrt{(1-y^2)}},$$

$$\int\sqrt{(1-x^2)}\,dy = y\sqrt{(1-x^2)}+\int\frac{xy\,dx}{\sqrt{(1-x^2)}};$$

hence we get

$$x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}+\int xy\left\{\frac{dy}{\sqrt{(1-y^2)}}+\frac{dx}{\sqrt{(1-x^2)}}\right\}$$

= a constant, or, since the quantity under the sign of integration vanishes,

$$x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)} = \text{a constant} = z,$$

as we have $x=z$, when y vanishes, from the equation

$$f(x)+f(y)=f(z).$$

We see thus, that if $f(x)+f(y)=f(z)$, then

$$z = x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)},$$

which may be considered as the fundamental property of the function $\sin^{-1}x$.

14. To integrate $\dfrac{dx}{\sqrt{(x^2-a^2)}}$.

We have seen already, in Art. 6, that the substitution $x=2a^2z/(z^2+a^2)$ transforms the radical $\sqrt{(a^2-x^2)}$ to a rational

form. Hence, putting a^2/x for x, we see that $\sqrt{(x^2 - a^2)}/x$, and therefore $\sqrt{(x^2 - a^2)}$ is rationalized by assuming $x = \frac{1}{2}(z + a^2/z)$. Differentiating then this relation, we get $(z - x)\,dz = z\,dx$; but $z - x = \sqrt{(x^2 - a^2)}$; therefore

$$\frac{dx}{\sqrt{(x^2 - a^2)}} = \frac{dz}{z},$$

and $$\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \int \frac{dz}{z} = \log z = \log \{x + \sqrt{(x^2 - a^2)}\}. \quad (14)$$

Hence, changing the sign of a^2, we have

$$\int \frac{dx}{\sqrt{(x^2 + a^2)}} = \log \{x + \sqrt{(x^2 + a^2)}\}. \quad (15)$$

By the aid of (4) and the forms in the preceding Article we can evidently integrate the expression

$$\frac{dx}{\sqrt{(a + 2bx + cx^2)}},$$

where a, b, c are any constant quantities.

Putting $cx + b = z$, as in Art. 10, we find from (14) and (4),

$$\int \frac{dx}{\sqrt{(a + 2bx + cx^2)}} = \frac{1}{\sqrt{c}} \log (cx + b + \sqrt{\{c(a + 2bx + cx^2)\}}), \quad (16)$$

or $$\frac{1}{\sqrt{-c}} \cos^{-1} \left\{\frac{b + cx}{\sqrt{(b^2 - ac)}}\right\}, \quad (17)$$

according as c is positive or negative.

If the factors of the quantity under the radical are given and real, we can exhibit the integrals under certain simple forms.

15. To find the integral of $\dfrac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}$, let $x-\alpha=z^2$;

then $$\frac{dx}{\sqrt{(x-\alpha)}}=2dz;$$

hence $$\frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}=\frac{2dz}{\sqrt{(z^2+\alpha-\beta)}},$$

and $$\int\frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}=2\int\frac{dz}{\sqrt{(z^2+\alpha-\beta)}},$$

$$=2\log\{z+\sqrt{(z^2+\alpha-\beta)}\},\text{ from (14)},$$

or $$\int\frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}=2\log\{\sqrt{(x-\alpha)}+\sqrt{(x-\beta)}\}. \quad (18)$$

In this integral x is supposed to be always greater than both α and β, but if it be less, we find in the same way,

$$\int\frac{dx}{\sqrt{\{(\alpha-x)(\beta-x)\}}}=-2\log\{\sqrt{(\alpha-x)}+\sqrt{(\beta-x)}\}. \quad (19)$$

Again, to find the integral of $\dfrac{dx}{\sqrt{\{(\alpha-x)(x-\beta)\}}}$, we might put $x=\alpha\sin^2\phi+\beta\cos^2\phi$.

We get then

$$\alpha-x=(\alpha-\beta)\cos^2\phi,\quad x-\beta=(\alpha-\beta)\sin^2\phi,$$

$$dx=2(\alpha-\beta)\sin\phi\cos\phi\,d\phi;$$

hence $$\frac{dx}{\sqrt{\{(\alpha-x)(x-\beta)\}}}=2d\phi,$$

and $$\int\frac{dx}{\sqrt{\{(\alpha-x)(x-\beta)\}}}=2\phi=2\tan^{-1}\sqrt{\left(\frac{x-\beta}{\alpha-x}\right)}. \quad (20)$$

Examples.

1. $\displaystyle\int \frac{dx}{\sqrt{(x^2+ax)}} = \log\{x + \tfrac{1}{2}a + \sqrt{(x^2+ax)}\}.$

2. $\displaystyle\int \frac{dx}{\sqrt{(ax-x^2)}} = 2\sin^{-1}\sqrt{\left(\frac{x}{a}\right)}.$

3. $\displaystyle\int \frac{(l+mx)\,dx}{\sqrt{(a+2bx+cx^2)}} = \frac{m}{c}\sqrt{(a+2bx+cx^2)} + \frac{lc-mb}{c}\int\frac{dx}{\sqrt{(a+2bx+cx^2)}}.$

4. $\displaystyle\int \frac{dx}{\sqrt{\{(\alpha+\beta x)(\gamma+\delta x)\}}} = \frac{2}{\sqrt{(\beta\delta)}}\log\left\{\sqrt{\left(\frac{\alpha+\beta x}{\beta}\right)} + \sqrt{\left(\frac{\gamma+\delta x}{\delta}\right)}\right\},$

$$\text{or}\qquad \frac{2}{\sqrt{(-\beta\delta)}}\sin^{-1}\sqrt{\left\{\frac{\delta(\alpha+\beta x)}{\alpha\delta-\beta\gamma}\right\}}.$$

5. $\displaystyle\int \sqrt{\left(\frac{x}{1+x}\right)}\,dx = \sqrt{(x+x^2)} + \tfrac{1}{2}\log\{x + \tfrac{1}{2} + \sqrt{(x+x^2)}\}.$

6. $\displaystyle\int \frac{dx}{\sqrt{(1-x+x^2)}} = \log\{2x - 1 + 2\sqrt{(1-x+x^2)}\}.$

7. $\displaystyle\int \frac{x\,dx}{\sqrt{\{(a^2-x^2)(x^2-b^2)\}}} = \sin^{-1}\left\{\sqrt{\left(\frac{x^2-b^2}{a^2-b^2}\right)}\right\}.$

8. $\displaystyle\int \frac{dx}{x\sqrt{(x^2-a^2)}} = \int\frac{-d(x^{-1})}{\sqrt{(1-a^2x^{-2})}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right).$

9. $\displaystyle\int \frac{dx}{x\sqrt{(x^2+a^2)}} = \int\frac{-d(x^{-1})}{\sqrt{(1+a^2x^{-2})}} = -\frac{1}{a}\log\{ax^{-1} + \sqrt{(1+a^2x^{-2})}\}$

$$= \frac{1}{a}\log\left\{\frac{\sqrt{(a^2+x^2)}-a}{x}\right\}.$$

10. $\displaystyle\int \frac{dx}{x}\sqrt{(x^2-1)} = \int\frac{x\,dx}{\sqrt{(x^2-1)}} - \int\frac{dx}{x\sqrt{(x^2-1)}} = \sqrt{(x^2-1)} - \sec^{-1}x.$

11. $\displaystyle\int \frac{dx}{x}\sqrt{\left(\frac{x+1}{x-1}\right)} = \log\{x+\sqrt{(x^2-1)}\} - \sin^{-1}\left(\frac{1}{x}\right).$

12. $\displaystyle\int \frac{dx}{x\sqrt{\{(a^2-x^2)(x^2-b^2)\}}} = \frac{1}{ab}\tan^{-1}\left\{\frac{a}{b}\sqrt{\left(\frac{x^2-b^2}{a^2-x^2}\right)}\right\}.$

13. Show by the Integral Calculus that

$$\log\{x + \sqrt{(x^2 - 1)}\} = i \cos^{-1} x.$$

14. Given $$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0,$$

show, without making use of (5), that

$$1 - xy = c(x + y),$$

where c is a constant.

16. To integrate $$\frac{dx}{(x-a)\sqrt{(a + 2bx + cx^2)}},$$

putting $x = a + z$, we have

$$\int \frac{dx}{(x-a)\sqrt{(a+2bx+cx^2)}} = \int \frac{dz}{z\sqrt{(a' + 2b'z + cz^2)}}$$

$$= \int \frac{-d(z^{-1})}{\sqrt{(c + 2b'z^{-1} + a'z^{-2})}},$$

where $$a' = a + 2ba + ca^2, \quad b' = b + ca.$$

We see thus that the integral is reducible to (16) or (17), according as a' is positive or negative. For instance, if a' is positive we get

$$\int \frac{dx}{(x-a)\Delta} = \frac{1}{\sqrt{a'}} \log\left\{\frac{x-a}{a + bx + a(cx+b) + \Delta\sqrt{a'}}\right\}, \quad (21)$$

where $$\Delta^2 = a + 2bx + cx^2.$$

From this result we can obtain the integral of the expression $\dfrac{(lx+m)\,dx}{\{(x-a)^2 + \beta^2\}\Delta}$, by making use of imaginaries. Substituting $a + i\beta$ for a, if we put

$$a + 2ba + c(a^2 - \beta^2) = m^2 \cos 2\lambda,$$

$$2\beta(b + ca) = m^2 \sin 2\lambda,$$

$$a + bx + a(cx + b) + \Delta m \cos\lambda = A,$$

$$\beta(cx + b) + \Delta m \sin\lambda = B,$$

and make use of the identity

$$\log(A+iB)=\tfrac{1}{2}\log(A^2+B^2)+i\tan^{-1}\left(\frac{B}{A}\right),$$

we find

$$\int\frac{dx}{(x-a-i\beta)\Delta},\ \text{or}\int\frac{(x-a+i\beta)\,dx}{\{(x-a)^2+\beta^2\}\,\Delta}$$

$$=-\frac{1}{m}(\cos\lambda-i\sin\lambda)\left(\tfrac{1}{2}\log(A^2+B^2)-\tfrac{1}{2}\log\{(x-a)^2+\beta^2\}\right)$$

$$-\frac{1}{m}(\sin\lambda+i\cos\lambda)\left\{\tan^{-1}\left(\frac{B}{A}\right)+\tan^{-1}\left(\frac{\beta}{x-a}\right)\right\}.$$

Hence at once we get

$$\int\frac{(x-a)\,dx}{\{(x-a)^2+\beta^2\}\,\Delta}=\frac{\cos\lambda}{m}\log\left\{\frac{(x-a)^2+\beta^2}{A^2+B^2}\right\}$$

$$-\frac{\sin\lambda}{m}\left\{\tan^{-1}\left(\frac{B}{A}\right)+\tan^{-1}\left(\frac{\beta}{x-a}\right)\right\},\qquad(22)$$

$$\int\frac{dx}{\{(x-a)^2+\beta^2\}\,\Delta}=\frac{\sin\lambda}{m}\log\left\{\frac{A^2+B^2}{(x-a)^2+\beta^2}\right\}$$

$$-\frac{\cos\lambda}{m}\left\{\tan^{-1}\left(\frac{B}{A}\right)+\tan^{-1}\left(\frac{\beta}{x-a}\right)\right\},\qquad(23)$$

upon which two integrals the integration of the expression given above can be evidently made to depend.

Examples.

1. $\displaystyle\int\frac{dx}{(a+bx)\sqrt{(1-x^2)}}=\frac{1}{\sqrt{(a^2-b^2)}}\sin^{-1}\left(\frac{b+ax}{a+bx}\right).$

2. $\displaystyle\int\frac{dx}{(a+bx)\sqrt{(1+x^2)}}=\frac{1}{\sqrt{(a^2+b^2)}}\log\left(\frac{a+bx}{b-ax+\sqrt{\{(a^2+b^2)(1+x^2)\}}}\right).$

3. $$\int \frac{dx}{(cx+b)\sqrt{(a+2bx+cx^2)}} = \frac{1}{mc}\log\left(\frac{\sqrt{(a+2bx+cx^2)}-m}{cx+b}\right),$$

if $ac - b^2 = m^2c$; but if $ac - b^2 = -n^2c$, the integral is

$$\frac{1}{nc}\tan^{-1}\left\{\frac{1}{n}\sqrt{(a+2bx+cx^2)}\right\}.$$

4. $$\int \frac{dx}{(a+x)\sqrt{(a^2-x^2)}} = -\frac{1}{a}\sqrt{\left(\frac{a-x}{a+x}\right)}.$$

5. $$\int \frac{dx}{(a+bx)\sqrt{(a+2bx+cx^2)}} = \frac{1}{ma}\log\left\{\frac{a+bx}{\sqrt{(a+2bx+cx^2)}-m}\right\},$$

if $ac - b^2 = m^2a$; but if $ac - b^2 = -n^2a$, the integral is

$$\frac{1}{na}\tan^{-1}\left(\frac{nx}{\sqrt{(a+2bx+cx^2)}}\right).$$

6. $$\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{\sqrt{(1-x^2)}}\right).$$

7. $$\int \frac{dx}{(x+5)\sqrt{\{(x+1)(x+4)\}}} = \tfrac{1}{2}\log\left\{\frac{2\sqrt{(x+4)}+\sqrt{(x+1)}}{2\sqrt{(x+4)}-\sqrt{(x+1)}}\right\}.$$

8. $$\int \frac{dx}{(x+3)\sqrt{\{(6-x)(x-1)\}}} = \tfrac{1}{3}\tan^{-1}\sqrt{\left\{\frac{3}{2}\left(\frac{x-1}{6-x}\right)\right\}}.$$

9. $$\int \frac{dx}{(1+x)\sqrt{(1+3x)}} = \sqrt{2}\tan^{-1}\sqrt{\left(\frac{1+3x}{2}\right)}.$$

10. $$\int \frac{dx}{(a+bx^2)\sqrt{(1-x^2)}} = \frac{1}{ma}\tan^{-1}\frac{mx}{\sqrt{(1-x^2)}}$$

if $a + b = m^2a$; but if $a + b = -n^2a$, the integral is

$$\frac{1}{2na}\log\left\{\frac{\sqrt{(1-x^2)}+nx}{\sqrt{(1-x^2)}-nx}\right\}.$$

11. $$\int \frac{dx}{(x-\alpha)\sqrt{\{(x-\alpha)(x-\beta)\}}} = 2(\beta-\alpha)\left\{\frac{x-\beta}{x-\alpha}\right\}^{\frac{1}{2}}.$$

12. $\displaystyle\int \frac{dx}{\{(x-\alpha)(x-\beta)\}^{\frac{3}{2}}} = \frac{2(\alpha+\beta-2x)}{(\alpha-\beta)^2\sqrt{\{(x-\alpha)(x-\beta)\}}}.$

13. $\displaystyle\int \frac{(\lambda+\mu x)\,dx}{\{(x-\alpha)(x-\beta)\}^{\frac{3}{2}}} = \frac{2\lambda(\alpha+\beta-2x)+2\mu\{2\alpha\beta-(\alpha+\beta)x\}}{(\alpha-\beta)^2\sqrt{\{(x-\alpha)(x-\beta)\}}}.$

14. $\displaystyle\int \frac{dx}{\{(a+2bx+cx^2)\}^{\frac{3}{2}}} = \frac{b+cx}{(ac-b^2)\sqrt{(a+2bx+cx^2)}}.$

17. The method of integration by parts, that is, the use of the formula given at the end of Art. 5, enables us to obtain very readily a great number of integrals which it would not be easy to determine otherwise. This is particularly the case when the expression to be integrated involves both algebraic and circular or logarithmic functions, as we can then frequently make the given integral depend upon another, in which the transcendental function disappears under the sign of integration. The advantage of this method can be best exhibited by applying it to the few following cases:—

To integrate $\sqrt{(x^2+a^2)}\,dx$,

let $\quad u = \sqrt{(x^2+a^2)}, \quad v = x, \quad$ in the formula

$$\int u\,dv = uv - \int v\,du.$$

We get then

$$\int\sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} - \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}};$$

but we have

$$\int\sqrt{(x^2+a^2)}\,dx = \int\frac{a^2\,dx}{\sqrt{(x^2+a^2)}} + \int\frac{x^2\,dx}{\sqrt{(x^2+a^2)}};$$

therefore, by addition, we get

$$2\int\sqrt{(x^2+a^2)}\,dx = x\sqrt{(x^2+a^2)} + a^2\int\frac{dx}{\sqrt{(x^2+a^2)}};$$

or finally, from (15),

$$\int\sqrt{(x^2+a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2\log\{x+\sqrt{(x^2+a^2)}\}. \quad (24)$$

Again, to determine the integral of $\log x dx$, putting $u = \log x$, $v = x$, we get

$$\int \log x dx = x \log x - \int dx = x (\log x - 1).$$

In the same way we find

$$\int \tan^{-1} x dx = x \tan^{-1} x - \log \sqrt{(1 + x^2)}.$$

We may frequently determine the value of an integral by repeating the process of integration by parts, as, for instance, to find the integral of $e^{ax} \sin mx\, dx$, we may take

$$u = \sin mx, \quad v = e^{ax}/a,$$

when we get

$$\int e^{ax} \sin mx\, dx = \frac{1}{a} e^{ax} \sin mx - \frac{m}{a} \int e^{ax} \cos mx\, dx,$$

and in the same way,

$$\int e^{ax} \cos mx\, dx = \frac{1}{a} e^{ax} \cos mx + \frac{m}{a} \int e^{ax} \sin mx\, dx.$$

Hence, substituting for $\int e^{ax} \cos mx\, dx$, and solving for $\int e^{ax} \sin mx\, dx$, we obtain

$$\int e^{ax} \sin mx\, dx = e^{ax} \frac{(a \sin mx - m \cos mx)}{a^2 + m^2}. \qquad (25)$$

In like manner we get

$$\int e^{ax} \cos mx\, dx = e^{ax} \frac{(a \cos mx + m \sin mx)}{a^2 + m^2}. \qquad (26)$$

It may be observed that the two latter integrals can be obtained at once by putting $a + im$ for a in (8), and equating the real and imaginary parts on both sides of the resulting equation.

Examples.

1. $\int \sqrt{(a^2 - x^2)}\, dx = \frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right).$

2. $\int \sqrt{(a + 2bx + cx^2)}\, dx = \frac{1}{2c}(cx + b)\sqrt{(a + 2bx + cx^2)}$

$$+ \frac{ac - b^2}{2c} \int \frac{dx}{\sqrt{(a + 2bx + cx^2)}}.$$

3. $\int (a + bx^2)^{\frac{3}{2}} dx = \frac{1}{4} x (a + bx^2)^{\frac{3}{2}} + \frac{3}{4} a \int (a + bx^2)^{\frac{1}{2}} dx.$

4. $\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{(1 - x^2)}.$

5. $\int x \sin^{-1} x\, dx = \frac{1}{4}(2x^2 - 1) \sin^{-1} x + \frac{1}{4} x \sqrt{(1 - x^2)}.$

6. $\int \log \{\sqrt{(x - \alpha)} + \sqrt{(x - \beta)}\} dx = \frac{1}{2}(2x - \alpha - \beta) \log \{\sqrt{(x - \alpha)} + \sqrt{(x - \beta)}\}$

$$- \frac{1}{2}\sqrt{\{(x - \alpha)(x - \beta)\}}.$$

7. $\int x^2 \tan^{-1} x\, dx = \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{3} \log \sqrt{(1 + x^2)}.$

8. $\int x^2 \cos x\, dx = (x^2 - 2) \sin x + 2x \cos x.$

$\frac{1}{4}(x^2 - x \sin 2x - \frac{1}{2} \cos 2x)$

9. $\int x \sin^2 x\, dx = \cancel{\frac{1}{4}(x^2 + x \cos 2x - \sin x \cos x)}.$

10. $\int \frac{\tan^{-1} x\, dx}{(1 + x^2)^{\frac{3}{2}}} = \frac{1 + x \tan^{-1} x}{\sqrt{(1 + x^2)}}.$

11. $\int x^m \log x\, dx = \frac{x^{m+1}}{m + 1} \log \left(\frac{x}{e^{m+1}}\right).$

12. $\int x^3 e^x dx = e^x (x^3 - 3x^2 + 6x - 6).$

18. In connexion with the process of integration by parts we give here a general formula, which will be shown to be of considerable use in the evaluation of integrals.

If we put

$$\Theta = u\frac{d^n v}{dx^n} - \frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \ldots + (-1)^n\frac{d^n u}{dx^n}v,$$

we shall have

$$\int u\frac{d^{n+1}v}{dx^{n+1}}dx = \Theta - (-1)^n\int v\frac{d^{n+1}u}{dx^{n+1}}dx. \qquad (27)$$

For, if we differentiate this equation, it becomes

$$u\frac{d^{n+1}v}{dx^{n+1}} = \frac{d\Theta}{dx} - (-1)^n v\frac{d^{n+1}u}{dx^{n+1}},$$

or

$$\frac{d\Theta}{dx} = u\frac{d^{n+1}v}{dx^{n+1}} + (-1)^n v\frac{d^{n+1}u}{dx^{n+1}},$$

which can be verified at once, as we have

$$\frac{d\Theta}{dx} = \frac{du}{dx}\frac{d^n v}{dx^n} - \frac{d^2u}{dx^2}\frac{d^{n-1}v}{dx^{n-1}} + \ldots + (-1)^n\frac{d^{n+1}u}{du^{n+1}}v$$

$$+ u\frac{d^{n+1}v}{dx^{n+1}} - \frac{du}{dx}\frac{d^n v}{dx^n} + \ldots + (-1)^n\frac{d^n u}{dx^n}\frac{dv}{dx};$$

but all the terms on the right-hand side of this equation destroy one another, except the last of the first line and the first of the second line, which gives the result stated above.

As an example, we may make use of (27) to determine the integral $\int e^{ax}f(x)\,dx$, where $f(x)$ is an integral polynomial expression in x of degree n. Taking then $u = f(x)$, $v = e^{ax}$, we have

$$\frac{d^{n+1}v}{dx^{n+1}} = a^{n+1}e^{ax}, \quad \frac{d^{n+1}u}{dx^{n+1}} = 0,$$

from which we infer

$$a^{n+1}\int e^{ax}f(x)\,dx = \Theta;$$

or, putting for Θ its value,

$$\int e^{ax}f(x)\,dx = \frac{1}{a}e^{ax}\left\{f(x) - \frac{1}{a}f'(x) + \ldots + \frac{(-1)^n}{a^n}f^{(n)}(x)\right\}. \quad (28)$$

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In like manner we can obtain the integrals

$$\int f(x) \cos ax\,dx = P \cos ax + Q \sin ax, \tag{29}$$

$$\int f(x) \sin ax\,dx = P \sin ax - Q \cos ax, \tag{30}$$

where
$$P = \frac{1}{a^2} f'(x) - \frac{1}{a^4} f'''(x) + \&c.,$$

$$Q = \frac{1}{a} f(x) - \frac{1}{a^3} f''(x) + \&c.$$

19. We now proceed to consider some integrals of differentials involving circular functions of the variable. These are, in general, reducible to integrals of algebraic expressions already given, but we shall here consider them separately.

To find the integral of $\dfrac{d\theta}{\sin\theta}$,
we have

$$\int \frac{d\theta}{\sin\theta} = \int \frac{d\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \int \frac{\sec^2\frac{1}{2}\theta\,d\theta}{2\tan\frac{1}{2}\theta} = \int \frac{d\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\theta}$$
$$= \log\tan\tfrac{1}{2}\theta. \tag{31}$$

Or thus:

$$\int \frac{d\theta}{\sin\theta} = \int \frac{\sin\theta\,d\theta}{\sin^2\theta} = -\int \frac{d(\cos\theta)}{1-\cos^2\theta} = -\tfrac{1}{2}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$
$$= \log\tan\tfrac{1}{2}\theta.$$

Similarly we find

$$\int \frac{d\theta}{\cos\theta} = \log(\sec\theta + \tan\theta), \quad \text{or} \quad \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right). \tag{32}$$

Again, to integrate $\dfrac{d\theta}{a + b\cos\theta}$,

we have

$$\int \frac{d\theta}{a + b\cos\theta} = \int \frac{d\theta}{(a+b)\cos^2\frac{1}{2}\theta + (a-b)\sin^2\frac{1}{2}\theta}$$

$$= \int \frac{\sec^2\frac{1}{2}\theta\, d\theta}{a + b + (a-b)\tan^2\frac{1}{2}\theta} = \int \frac{\frac{1}{2}d\tan\frac{1}{2}\theta}{a+b+(a-b)\tan^2\frac{1}{2}\theta}.$$

Hence, from (5), if $a > b$, we get

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan\frac{\theta}{2}\right\},$$

or

$$\frac{1}{\sqrt{(a^2 - b^2)}} \cos^{-1}\left\{\frac{b + a\cos\theta}{a + b\cos\theta}\right\}; \tag{33}$$

and, from (11), if $a < b$,

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{(b^2 - a^2)}} \log\left\{\frac{\sqrt{(b+a)} + \sqrt{(b-a)}\tan\frac{1}{2}\theta}{\sqrt{(b+a)} - \sqrt{(b-a)}\tan\frac{1}{2}\theta}\right\}. \tag{34}$$

Examples.

1. $\displaystyle\int \frac{d\theta}{\cos\theta - \cos\alpha} = \frac{1}{\sin\alpha}\log\left\{\frac{\sin\frac{\alpha+\theta}{2}}{\sin\frac{\alpha-\theta}{2}}\right\}.$

2. $\displaystyle\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{(b^2 - a^2)}}\log\left\{\frac{b + a\cos\theta + \sqrt{(b^2 - a^2)}\sin\theta}{a + b\cos\theta}\right\}.$

3. $\displaystyle\int \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{1}{\sqrt{(a^2 - b^2 - c^2)}}\cos^{-1}\left\{\frac{b^2 + c^2 + ab\cos\theta + ac\sin\theta}{(a + b\cos\theta + c\sin\theta)\sqrt{(b^2 + c^2)}}\right\}.$

4. $\displaystyle\int \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{1}{m}\log\left\{\frac{b^2 + c^2 + (ab - mc)\cos\theta + (ac + mb)\sin\theta}{a + b\cos\theta + c\sin\theta}\right\},$

where $$m^2 = b^2 + c^2 - a^2.$$

This and the preceding integral are reduced to (33) and (34) respectively by putting $b = e\cos\alpha$, $c = e\sin\alpha$.

5. $\displaystyle\int \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan\theta\right).$

6. $\displaystyle\int \frac{d\theta}{a^2 + b^2 + 2ab\cos\theta} = \frac{2}{a^2 - b^2}\tan^{-1}\left\{\frac{a-b}{a+b}\tan\frac{\theta}{2}\right\}.$

7. $\displaystyle\int \frac{d\theta}{a + b\sin\theta} = \frac{2}{\sqrt{(a^2 - b^2)}}\tan^{-1}\left\{\frac{b + a\tan\frac{\theta}{2}}{\sqrt{(a^2 - b^2)}}\right\},$

or $\displaystyle\frac{1}{\sqrt{(a^2 - b^2)}}\sin^{-1}\left\{\frac{b + a\sin\theta}{a + b\sin\theta}\right\},$ if $a > b$.

8. $\displaystyle\int \frac{d\theta}{a + b\sin\theta} = \frac{1}{\sqrt{(b^2 - a^2)}}\log\left\{\frac{b + a\tan\frac{1}{2}\theta - \sqrt{(b^2 - a^2)}}{b + a\tan\frac{1}{2}\theta + \sqrt{(b^2 - a^2)}}\right\},$ if $a < b$.

9. From the expression given by the Differential Calculus for the sine of the angle at which the radius vector meets a curve, namely, $r\,d\theta / ds$, show that, if θ, ϕ are the base angles of the triangle formed by the foci of an ellipse and a point on the curve, $\tan\frac{1}{2}\theta\tan\frac{1}{2}\phi =$ a constant.

10. $\displaystyle\int \frac{d\theta}{a + b\tan\theta} = \frac{b}{a^2 + b^2}\log(a\cos\theta + b\sin\theta) + \frac{a\theta}{a^2 + b^2}.$

11. $\displaystyle\int \frac{d\theta}{a\cos\theta + b\sin\theta} = \frac{1}{\sqrt{(a^2 + b^2)}}\log\left\{\frac{a\sin\theta - b\cos\theta + \sqrt{(a^2 + b^2)}}{a\cos\theta + b\sin\theta}\right\}.$

12. $\displaystyle\int \frac{(l + m\cos\theta + n\sin\theta)\,d\theta}{a\cos\theta + b\sin\theta} = \frac{(ma + nb)}{a^2 + b^2}\left\{\theta - \int\frac{l\,d\theta}{a\cos\theta + b\sin\theta}\right\}$

$$+ \frac{(mb - na)}{a^2 + b^2}\log(a\cos\theta + b\sin\theta).$$

20. In the case of integrals of algebraic differentials, it may be of interest to notice the forms which these expressions take when they are rendered homogeneous, that is, when we substitute y/x for x. Now, if we put y/x for x in $f(x)\,dx$, we get

$$\frac{1}{x^2}f\left(\frac{y}{x}\right)(x\,dy - y\,dx),$$

from which we see that if the expression $\phi(x, y)(x\,dy - y\,dx)$ is

capable of integration without assuming any relation between x and y, then $\phi(x, y)$, namely, the coefficient of $x\,dy - y\,dx$, must be a homogeneous function in x, y of the degree -2. The advantages which follow hence are those which have place in algebra from the use of homogeneous expressions, namely, symmetry of form and the applicability of the linear or homographic transformation. Suppose we transform x, y by the linear substitution, that is, let

$$x = \alpha x' + \beta y', \quad y = \alpha' x' + \beta' y', \tag{35}$$

then dx, dy are evidently transformed in the same way, and, therefore, we have

$$x\,dy - y\,dx = (\alpha\beta' - \beta\alpha')(x'dy' - y'dx'); \tag{36}$$

that is, $x\,dy - y\,dx$ is transformed into the product of the similar expression for the new variables by the determinant or modulus of transformation. If then we transform a differential expression $\phi(x, y)(x\,dy - y\,dx)$ by the linear substitution (35), we get $\psi(x', y')(x'dy' - y'dx')$ multiplied by the modulus of transformation. In this manner we can frequently simplify the expression to be integrated by a proper choice of the new variables, in the same way as this result is effected in the theory of binary quantics.

From this point of view two general forms of integrals representing the circular or logarithmic functions are—

$$\int \frac{x\,dy - y\,dx}{u}, \tag{37}$$

$$\int \frac{x\,dy - y\,dx}{v\sqrt{u}}, \tag{38}$$

where u is a quadratic, and v is linear in x, y.

We see then that one form cannot be reduced to the other by a linear transformation. The reduction, however, can be effected by a quadratic transformation, as will be proved in a subsequent chapter.

The integral (37) will evidently be logarithmic or circular according as the factors of u are real or imaginary. In the first case, if these factors are $lx + my$, $l'x + m'y$, and if we call them x', y', respectively, we get at once

$$\int \frac{x\,dy - y\,dx}{u} = \frac{1}{lm' - l'm} \log\left(\frac{y'}{x'}\right). \tag{39}$$

In the second case, we may put

$$u = (lx + my)^2 + (l'x + m'y)^2 = x'^2 + y'^2, \text{ say,}$$

and then we have

$$\int \frac{x\,dy - y\,dx}{u} = \frac{1}{lm' - l'm} \tan^{-1}\left(\frac{y'}{x'}\right). \tag{40}$$

21. As an application of the method of the preceding Article, we may investigate the value of the integral

$$\int \frac{pz + q}{\{(z - a)^2 + \beta^2\}} \frac{dz}{\sqrt{(a^2 - z^2)}}.$$

From Art. 6, we see that $\sqrt{(a^2 - z^2)}$ is rationalized by taking

$$z = \frac{2ax}{1 + x^2}, \quad \text{or} \quad \frac{2axy}{x^2 + y^2},$$

if we put y/x for x. With this substitution the given integral takes the form

$$\int \frac{R}{S} (x\,dy - y\,dx),$$

where R is a quadratic, and S a biquadratic expression in

x, y. Now the linear factors of S are evidently all imaginary, and, therefore, from algebraical considerations, S can be expressed in a single manner as the product of two real quadratic expressions, P, Q, say. Again, if J denote the Jacobian of P, Q, namely the quadratic

$$\frac{dP}{dx}\frac{dQ}{dy} - \frac{dP}{dy}\frac{dQ}{dx},$$

it is easy to see that, in general, we may write any arbitrary quadratic in the form $lP + mQ + nJ$, that is, we may assign the constants l, m, n, so that this expression may coincide with any given quadratic. Supposing then R to take this form, the integral given above becomes

$$\int \frac{lP + mQ + nJ}{PQ} (x\,dy - y\,dx),$$

or

$$l\int \frac{(x\,dy - y\,dx)}{Q} + m\int \frac{(x\,dy - y\,dx)}{P} + n\int \frac{J(x\,dy - y\,dx)}{PQ}. \tag{41}$$

Now, since the factors of P and Q are imaginary, the first two terms in (41) are, from (40), expressible by angles; and the third term is expressible by a logarithm, as we shall show now. We have

$$\begin{aligned} PdQ - QdP &= \tfrac{1}{2}\left(x\frac{dP}{dx} + y\frac{dP}{dy}\right)\left(\frac{dQ}{dx}dx + \frac{dQ}{dy}dy\right) \\ &\quad - \tfrac{1}{2}\left(x\frac{dQ}{dx} + y\frac{dQ}{dy}\right)\left(\frac{dP}{dx}dx + \frac{dP}{dy}dy\right) \\ &= \tfrac{1}{2}(x\,dy - y\,dx)\left(\frac{dP}{dx}\frac{dQ}{dy} - \frac{dP}{dy}\frac{dQ}{dx}\right) = \tfrac{1}{2}J(x\,dy - y\,dx). \end{aligned}$$

Thus we get

$$\frac{J(x\,dy - y\,dx)}{PQ} = \frac{2(P\,dQ - Q\,dP)}{PQ},$$

and consequently the third term of (41) is equal to

$$2n \log\left(\frac{Q}{P}\right).$$

22. In addition to the methods given in Art. 5 for the reduction of integrals to known forms, we may notice here the principle of differentiating or integrating under the sign of integration. Let a be any quantity involved in a function of x, which we may therefore denote by $f(x, a)$, and suppose we have any equation

$$\int f(x, a)\, dx = \phi(x, a),$$

then
$$f(x, a) = \frac{d\phi(x, a)}{dx},$$

and
$$\frac{df(x, a)}{da} = \frac{d^2\phi(x, a)}{da\, dx} = \frac{d^2\phi(x, a)}{dx\, da}.$$

Hence, integrating with regard to x, we get

$$\int \frac{df(x, a)}{da}\, dx = \frac{d\phi(x, a)}{da},$$

that is, if we have $\int u\, dx = v$,

then
$$\int \frac{du}{da}\, dx = \frac{dv}{da}, \tag{42}$$

where a is any quantity that is involved in u, and does not vary with x.

We may evidently repeat this process as often as we wish, so that we have

$$\int \frac{d^n u}{da^n}\, dx = \frac{d^n v}{da^n}. \tag{43}$$

In exactly the same way we can show that we may integrate with regard to a quantity a involved in u and v, that is, if, as before, $\int u dx = v$, then we have

$$\int \left(\int u\, da\right) dx = \int v\, da. \tag{44}$$

EXAMPLES.

1. $$\int \frac{dx}{(x^2+a)^2} = -\frac{d}{da}\int \frac{dx}{x^2+a} = -\frac{d}{da}\left(\frac{1}{\sqrt{a}}\tan^{-1}\frac{x}{\sqrt{a}}\right)$$
$$= \frac{1}{2a^{\frac{3}{2}}}\tan^{-1}\frac{x}{\sqrt{a}} - \frac{1}{2a}\,\frac{x}{x^2+a}.$$

2. $$\int \frac{dx}{(x^2-a)^2} = \frac{1}{4a^{\frac{3}{2}}}\log\left(\frac{x+\sqrt{a}}{x-\sqrt{a}}\right) + \frac{1}{2a}\,\frac{x}{x^2-a}.$$

3. $$\int \frac{dx}{(x^2+a)^{\frac{5}{2}}} = -\frac{2}{3}\frac{d}{da}\int \frac{dx}{(x^2+a)^{\frac{3}{2}}} = \frac{x\,(3a+2x^2)}{3a^2(x^2+a)^{\frac{3}{2}}}.$$

4. $$\int x^m \log x\,dx = \frac{d}{dm}\int x^m dx = \frac{x^{m+1}}{m+1}\left\{\log x - \frac{1}{m+1}\right\}.$$

5. $$\int x^m (\log x)^2 dx = \frac{x^{m+1}}{(m+1)^3} + \frac{x^{m+1}}{m+1}\left(\log x - \frac{1}{m+1}\right)^2.$$

6. $$\int \frac{d\theta}{(a+b\cos\theta)^2} = -\frac{d}{da}\int \frac{d\theta}{a+b\cos\theta}$$
$$= \frac{2a}{(a^2-b^2)^{\frac{3}{2}}}\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\tfrac{1}{2}\theta\right\} - \frac{b\sin\theta}{(a^2-b^2)(a+b\cos\theta)}.$$

7. $$\int x^2 e^{ax}\,dx = \frac{d^2}{da^2}\int e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2).$$

8. $$\int \theta\cos m\theta\,d\theta = \frac{d}{dm}\int \sin m\theta\,d\theta = \frac{1}{m^2}\cos m\theta + \frac{\theta}{m}\cos m\theta.$$

23. We have now enumerated and exemplified most of the methods which are generally adopted for the reduction of integrals to the elementary circular and logarithmic forms. The different methods may sometimes be made use of to obtain the same integral, and must, of course, then give the same results, the identification of which may often serve to throw additional light on the processes employed.

The reduction to the forms just referred to is always possible, when the given differential coefficient is a rational function of x and the square root of an expression which does not involve powers of x beyond the second. The most important methods in this case are (2), (3), and (4) of Art. 5, and to each of these we propose to devote a chapter.

The same methods of reduction are also applicable, if the expression under the square root involves higher powers than the second. When this expression is of the third or fourth degree, the integral can be made to depend upon three fundamental forms, namely, the three kinds of elliptic integrals, some elementary properties of which we shall consider in a separate chapter.

24. We can give a geometrical illustration of the preceding remarks. Let x, y be the Cartesian co-ordinates of a point on a conic, then if $\phi(x, y)$ denotes any rational function of x, y, the integral

$$\int \phi(x, y)\, dx \tag{45}$$

depends upon the elementary circular and logarithmic transcendents. This can be seen to be the case, by supposing the conic to be written

$$y^2 = ax^2 + 2bx + c,$$

when the integral evidently assumes the form referred to in the preceding Article. Again, similarly, the integral (45) will depend upon elliptic integrals, if the point x, y lies on a general curve of the third degree, and, more generally, if the point lies on a curve the n^{th} degree, (45) will depend upon certain fundamental forms of the integrals which have been called Abelian. We shall return hereafter to the consideration of transcendents from this point of view.

EXAMPLES.

1. $\int \frac{dx}{(1+x)\sqrt{x}} = \tan^{-1} \frac{2\sqrt{x}}{1-x}.$

2. $\int \frac{dx}{\sqrt{(x-x^2)}} = \sin^{-1}\{2\sqrt{(x-x^2)}\}.$

3. $\int \frac{dx}{\sqrt{(1-x^2)}} = \frac{1}{3}\sin^{-1}(3x-4x^3).$

4. $\int \frac{dx}{e^x+e^{-x}} = \tan^{-1}(e^x).$

5. $\int \frac{dx}{x(\log x)^n} = -\frac{1}{n-1}\,\frac{1}{(\log x)^{n-1}}.$

6. $\int \frac{dx}{x\log x} = \log(\log x).$

7. $\int \frac{x^2\,dx}{(x\sin x+\cos x)^2} = \frac{\sin x - x\cos x}{x\sin x+\cos x}.$

8. $\int \frac{x^2\,dx}{(x\cos x-\sin x)^2} = \frac{x\sin x+\cos x}{x\cos x-\sin x}.$

9. $\int \frac{x^2\,dx}{\{(ax-b)\sin x+(a+bx)\cos x\}^2} = \frac{(x\sin x+\cos x)}{b\{(ax-b)\sin x+(a+bx)\cos x\}}.$

10. $\int \frac{x^2\,dx}{(x+a)^3} = \frac{a}{2}\,\frac{(3a+4x)}{(x+a)^2} + \log(x+a).$

11. $\int \frac{x^3\,dx}{(a+bx)} = \frac{b^2}{a^3}\log\left(\frac{x}{a+bx}\right) - \frac{1}{2a^3}\left(\frac{a}{x}-b\right)^2.$

12. $\int \frac{x^3\,dx}{(a+bx)^4} = -\frac{x}{(a+bx)^2} - \frac{a^2}{3b(a+bx)^3}.$

13. $\int \frac{x^3\,dx}{a+bx} = \frac{1}{6b^4}(a+bx)(2b^2x^2-5abx+11a^2) - \frac{a^3}{b^4}\log(a+bx).$

14. $\int \frac{dx}{\sqrt{(2+2x-4x^2)}} = \frac{1}{2}\sin^{-1}\left(\frac{4x-1}{3}\right).$

15. $$\int \frac{dx}{\sqrt{(2+2x+4x^2)}} = \tfrac{1}{2}\log\{4x+1+2\sqrt{(2+2x+4x^2)}\}.$$

16. $$\int \frac{dx}{\sqrt{\{(x-1)(5-x)\}}} = 2\sin^{-1}\{\tfrac{1}{2}\sqrt{(x-1)}\}.$$

17. $$\int \frac{dx}{x\sqrt{(7x^2+6x-1)}} = \sin^{-1}\left\{\frac{3x-1}{4x}\right\}.$$

18. $$\int \frac{dx}{x\sqrt{(3x^2+4x-4)}} = \tfrac{1}{2}\sin^{-1}\left(\frac{x-2}{2x}\right).$$

19. $$\int \sqrt{\left(\frac{x+3}{x+1}\right)}\,dx = \sqrt{\{(x+1)(x+3)\}}.$$
$$+\log\{x+2+\sqrt{\{(x+1)(x+3)\}}\}.$$

20. $$\int \frac{dx}{x\sqrt{(x^n-1)}} = \frac{2}{n}\cos^{-1}\left(x^{-\frac{n}{2}}\right).$$

21. $$\int \frac{dx}{x}\sqrt{(x^n-1)} = \frac{2}{n}\sqrt{(x^n-1)} - \frac{2}{n}\sin^{-1}\left(x^{-\frac{n}{2}}\right).$$

22. $$\int \frac{x\sqrt{(5+2x^2)}\,dx}{x^2+3} = \sqrt{(5+2x^2)} - \tan^{-1}\{\sqrt{(5+2x^2)}\}.$$

23. $$\int \frac{x\,dx}{(x^2+3)\sqrt{(5+2x^2)}} = \tan^{-1}\{\sqrt{(5+2x^2)}\}.$$

24. $$\int \frac{dx}{(x+1)\sqrt{(8x-x^2)}} = \tfrac{2}{3}\tan^{-1}\left\{3\sqrt{\left(\frac{x}{8-x}\right)}\right\}.$$

25. $$\int \frac{(p+qx)\,dx}{(2+6x+5x^2)^{\frac{3}{2}}} = \frac{3p-2q+(5p-3q)x}{\sqrt{(2+6x+5x^2)}}.$$

26. $$\int \frac{dx}{x^4\sqrt{(x^2-a^2)}} = \frac{(2x^2+a^2)}{3a^4x^3}\sqrt{(x^2-a^2)}.$$

27. $$\int \frac{x^7\,dx}{\sqrt{(1+x^2)}} = \sqrt{(1+x^2)}\{x^2-\tfrac{3}{5}(1+x^2)^2+\tfrac{1}{7}(1+x^2)^3\}.$$

28. $$\int \sqrt{\{(a-x)(x-b)\}}\,dx = \tfrac{1}{4}(2x-a-b)\sqrt{\{(a-x)(x-b)\}}$$
$$+\tfrac{1}{8}(a-b)^2\sin^{-1}\left\{\frac{2x-a-b}{a-b}\right\}.$$

29. $\displaystyle\int \frac{dx}{x^2} \tan^{-1} x = -\frac{1}{x} \tan^{-1} x + \log \left\{\frac{x}{\sqrt{(1+x^2)}}\right\}.$

30. $\displaystyle\int \frac{dx}{x^2} \sin^{-1} x = -\frac{1}{x} \sin^{-1} x + \log \left\{\frac{x}{1+\sqrt{(1-x^2)}}\right\}.$

31. $\displaystyle\int x^2 \log(x^2-1)\, dx = \tfrac{1}{3} x^3 \log(x^3-1) - \tfrac{2}{3} x - \tfrac{2}{9} x^3 - \tfrac{1}{3} \log\left\{\frac{x-1}{x+1}\right\}.$

32. $\displaystyle\int \tan^4\theta\, d\theta = \tfrac{1}{3}\tan^3\theta - \tan\theta + \theta.$

33. $\displaystyle\int \cos\theta \sqrt{(1-e^2\sin^2\theta)}\, d\theta = \tfrac{1}{2}\sin\theta\sqrt{(1-e^2\sin^2\theta)} + \tfrac{1}{2}e^{-1}\sin^{-1}(e\sin\theta).$

34. $\displaystyle\int \sin\theta \sqrt{(1-e^2\sin^2\theta)}\, d\theta = -\tfrac{1}{2}\cos\theta\sqrt{(1-e^2\sin^2\theta)}$

$$-\frac{(1-e^2)}{2e}\log\{\cos\theta+\sqrt{(1-e^2\sin^2\theta)}\}.$$

35. $\displaystyle\int \frac{\tan\theta\, d\theta}{\sqrt{(a^2+b^2\tan^2\theta)}} = \frac{1}{\sqrt{(b^2-a^2)}}\cos^{-1}\left\{\cos\theta\sqrt{\left(1-\frac{a^2}{b^2}\right)}\right\},$

or $$-\frac{1}{\sqrt{(a^2-b^2)}}\log\{\sqrt{(a^2-b^2)}\cos\theta+\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}\}.$$

36. $\displaystyle\int \frac{d\theta}{\sqrt{(a^2+b^2\tan^2\theta)}} = \frac{1}{\sqrt{(a^2-b^2)}}\sin^{-1}\left\{\sin\theta\sqrt{\left(1-\frac{b^2}{a^2}\right)}\right\},$

or $$\frac{1}{\sqrt{(b^2-a^2)}}\log\{\sqrt{(b^2-a^2)}\sin\theta+\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}\}.$$

37. $\displaystyle\int \frac{d\theta}{\sin\theta}\sqrt{(a^2+b^2\sin^2\theta)} = b\cos^{-1}\left\{\frac{b\cos\theta}{\sqrt{(a^2+b^2)}}\right\}$

$$-a\log\{a\cot\theta+\sqrt{(b^2+a^2\operatorname{cosec}^2\theta)}\}.$$

38. $\displaystyle\int (\sin\theta)^{n-1}\sin(n+1)\theta\, d\theta = \int(\sin\theta)^{n-1}(\sin n\theta\cos\theta+\cos n\theta\sin\theta)\,d\theta$

$$=\frac{1}{n}\int\{\sin n\theta\, d(\sin\theta)^n+(\sin\theta)^n\, d(\sin n\theta)\} = \frac{1}{n}\sin n\theta\,(\sin\theta)^n.$$

39. $\displaystyle\int (\sin\theta)^{n-1}\cos(n+1)\theta\, d\theta = \frac{1}{n}\cos n\theta\,(\sin\theta)^n.$

40. Given
$$U = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
$$H = (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2,$$
$$G = (a^2d + 2b^3 - 3abc)x^3 + 3(abd + b^2c - 2ac^2)x^2y$$
$$+ 3(2db^2 - acd - bc^2)xy^2 + (3bcd - 2c^3 - ad^2)y^3,$$
show that
$$\int \frac{H^2}{U^2}(x\,dy - y\,dx) = -\frac{G}{6U}.$$

41. With the same notation as in the preceding example, show that
$$\int \frac{H^2(x\,dy - y\,dx)}{UG} = \frac{1}{6}\log\left(\frac{U}{G}\right).$$

42. If P, Q are two quadratics in x, y, and J has the same meaning as in Art. 21, show that
$$\int \frac{J(x\,dy - y\,dx)}{P^{1+m}\,Q^{1-m}} = \frac{2}{m}\left(\frac{Q}{P}\right)^m.$$

CHAPTER II.

INTEGRATION OF RATIONAL FUNCTIONS.

25. We propose to consider in this chapter the integrals of $f(x)\,dx$, where $f(x)$ is a rational algebraic function of x. The most general form of such a function is evidently a fraction, namely, the ratio of two expressions, P, Q, say, which can be written

$$P = p_0x^m + p_1x^{m-1} + \dots + p_m,$$

$$Q = q_0x^n + q_1x^{n-1} + \dots + q_n,$$

where m, n are positive integers, and $p_0, p_1, \dots p_m, q_0, q_1, \dots q_n$ are constants.

If the degree of the numerator be equal to, or greater than, that of the denominator, the fraction P/Q may be reduced by division to the sum of an integral expression and another fraction in which the numerator is of a lower degree than the denominator. The first part can be integrated at once, so that we may confine our special attention to the case when m is less than n. The method of integration adopted in this case is (2) of Art. 5, namely, by decomposition of the given expression into fractions with simpler denominators, which are called partial fractions. We have had already a very simple example of the application of this method in Art. 9.

26. Supposing the fraction P/Q to be denoted by $\phi(x)/f(x)$, we know, from algebraical considerations, that $f(x)$ can be expressed as the product of a series of factors of the form

$$x-a, \quad (x-b)^p, \quad (x^2-2\alpha x+\alpha^2+\beta^2), \quad (x^2-2\alpha' x+\alpha'^2+\beta'^2)^q.$$

The simplest method will be then to consider these four different kinds of factors separately, and this is what we proceed to do in the following Articles:—

27. First, let all the factors of $f(x)$ be of the form $x-a$, that is, suppose that all the roots of the equation $f(x)=0$ are real and unequal, then we have

$$f(x)=(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n), \tag{1}$$

where $a_1, a_2, \ldots a_n$ are the n roots of $f(x)$.

We may put now

$$\frac{\phi(x)}{f(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}, \tag{2}$$

where $A_1, A_2 \ldots A_n$ are constants; for, if we multiply both sides of this equation by $f(x)$, and then equate like powers of x, we obtain n linear equations for the determination of the n quantities $A_1, A_2, \ldots A_n$. The simplest method, however, of finding these quantities is as follows:—

From (1) and (2) we have

$$\phi(x)=A_1(x-a_2)(x-a_3)\ldots(x-a_n)+A_2(x-a_1)(x-a_3)\ldots(x-a_n)$$
$$+\ldots+A_n(x-a_1)(x-a_2)\ldots(x-a_{n-1});$$

but this identity is, by hypothesis, to hold for all values of x; therefore, putting $x=a_r$, where a_r is one of the roots of $f(x)$,

all the terms except the coefficient of A_r vanish, and we get

$$A_r = \frac{\phi(a_r)}{f'(a_r)}, \tag{3}$$

observing that $(a_r - a_1)(a_r - a_2) \ldots (a_r - a_n) = f'(a_r)$.

Hence, giving to r all values from 1 to n inclusive, we obtain

$$\frac{\phi(x)}{f(x)} = \frac{\phi(a_1)}{f'(a_1)}\frac{1}{x-a_1} + \frac{\phi(a_2)}{f'(a_2)}\frac{1}{x-a_2} + \ldots + \frac{\phi(a_n)}{f'(a_n)}\frac{1}{x-a_n}, \tag{4}$$

and, therefore,

$$\int\frac{\phi(x)}{f(x)}dx = \frac{\phi(a_1)}{f'(a_1)}\log(x-a_1) + \frac{\phi(a_2)}{f'(a_2)}\log(x-a_2) + \ldots$$
$$+ \frac{\phi(a_n)}{f'(a_n)}\log(x-a_n),$$

which we may write

$$\int\frac{\phi(x)}{f(x)}dx = \Sigma_n{}^1\frac{\phi(a_r)}{f'(a_r)}\log(x-a_r), \tag{5}$$

where the summation refers to r.

In this result x is supposed to exceed the greatest of the roots a_1, a_2, &c.; but if it be less than any one, a_s say, of these quantities, $\log(x - a_s)$ must of course be replaced by $\log(a_s - x)$.

From this investigation we see that the integral contains a single term of the form

$$\frac{\phi(a_r)}{f'(a_r)}\log(x-a_r),$$

corresponding to a root a_r of $f(x)$. But this expression remains real and determinate, if any of the other roots are

imaginary or become equal to one another, provided a_r itself is a real determinate quantity and not a multiple root. We see thus that (5) gives us all the terms in the integral arising from real unequal roots of $f(x)$, the summation in this case extending to these roots only, and the parts depending on the other roots being determined by a different process.

It may be observed that if the degree of $\phi(x)$ exceed $n-1$, the partial fractions involved are still of the same form.

Let $\phi(x)/f(x)$ in this case be put equal to $\psi(x)+\phi_1(x)/f(x)$, where $\psi(x)$ is an integral expression, and $\phi_1(x)$ is of the degree $n-1$, then

$$\phi(x)=\psi(x)f(x)+\phi_1(x),$$

and therefore

$$\phi(a_r)=\phi_1(a_r),$$

which establishes the remark just made.

Examples.

1. $$\int \frac{dx}{(x-a)(x-b)(2x-a-b)}=\frac{1}{(a-b)^2}\log\left\{\frac{(x-a)(x-b)}{(2x-a-b)^2}\right\}.$$

2. $$\int \frac{dx}{(x-a)(x-b)(3x-2a-b)}=\frac{1}{2(a-b)^2}\log\left\{\frac{(x-a)^2(x-b)}{(3x-2a-b)^3}\right\}.$$

3. $$\int \frac{x\,dx}{x^3-6x^2+11x-6}=\int \frac{x\,dx}{(x-1)(x-2)(x-3)}$$

$$=\tfrac{1}{2}\int\left\{\frac{1}{x-1}+\frac{3}{x-3}-\frac{4}{x-2}\right\}dx$$

$$=\tfrac{1}{2}\log\left\{\frac{(x-1)(x-3)^3}{(x-2)^4}\right\}.$$

4. $$\int \frac{x^3\,dx}{(x-1)(x-2)(x-3)}=x+\tfrac{1}{2}\log\left\{\frac{(x-1)(x-3)^{27}}{(x-2)^{16}}\right\}.$$

5. $$\int \frac{dx}{x^3 - 7x + 6} = \frac{1}{20} \log \left\{ \frac{(x-2)^4 (x+3)}{(x-1)^5} \right\}.$$

6. $$\int \frac{x^2\,dx}{x^3 - 7x + 6} = \tfrac{1}{3} \int \frac{(3x^2 - 7 + 7)\,dx}{x^3 - 7x + 6}$$

$$= \tfrac{1}{3} \log (x^3 - 7x + 6) + \tfrac{7}{60} \log \left\{ \frac{(x-2)^4 (x+3)}{(x-1)^5} \right\}.$$

7. $$\int \frac{dx}{x(1-x^2)} = \log \left\{ \frac{x}{\sqrt{(1-x^2)}} \right\}.$$

8. $$\int \frac{x^4\,dx}{x^3 - 13x + 12}.$$

The factors of $x^3 - 13x + 12$ are $x - 1$, $x - 3$, $x + 4$, so that we may assume

$$\frac{x^4}{x^3 - 13x + 12} = x + \cancel{k} + \frac{A_1}{x-1} + \frac{A_2}{x-3} + \frac{A_3}{x+4}.$$

Multiplying up, we find, from the coefficients of x^3, $\cancel{k = 13}$, and we have

$$A_1 = -\tfrac{1}{10}, \quad A_2 = \frac{\overset{81}{\cancel{27}}}{14}, \quad A_3 = \tfrac{256}{35}.$$

Hence, we get

$$\int \frac{x^4\,dx}{x^3 - 13x + 12} = \tfrac{1}{2}x^2 + \cancel{13x} - \tfrac{1}{10} \log (x-1) + \frac{\overset{81}{\cancel{27}}}{14} \log (x-3) + \tfrac{256}{35} \log (x+4).$$

9. $$\int \frac{dx}{x^3 - 37x + 84} = \tfrac{1}{10} \log \left(\frac{x+7}{x-3} \right) + \tfrac{1}{11} \log \left(\frac{x-4}{x+7} \right).$$

10. $$\int \frac{x^3\,dx}{x^3 - 37x + 84} = x - \tfrac{27}{10} \log (x-3) + \tfrac{64}{11} \log (x-4) - \tfrac{343}{110} \log (x+7).$$

11. $$\int \frac{x^4\,dx}{(x-1)(x-2)} = \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 7x + \log \left\{ \frac{(x-2)^{16}}{x-1} \right\}.$$

12. $$\int \frac{dx}{x(x^2+\alpha)(x^2+\beta)} = \tfrac{1}{2} \int \frac{dz}{z(z+\alpha)(z+\beta)},$$ putting $x^2 = z$.

Hence we get

$$\int \frac{dx}{x(x^2+\alpha)(x^2+\beta)} = \frac{1}{2\beta(\alpha-\beta)} \log \left(\frac{x^2}{x^2+\beta} \right) - \frac{1}{2\alpha(\alpha-\beta)} \log \left(\frac{x^2}{x^2+\alpha} \right).$$

13. $\int \frac{dx}{(x^2-1)(x^2-4)} = \frac{1}{3}\int \left\{\frac{1}{x^2-4} - \frac{1}{x^2-1}\right\} dx$

$$= \frac{1}{12}\log\left\{\frac{(x-2)(x+1)^2}{(x+2)(x-1)^2}\right\}.$$

14. Show that

$$\int \frac{dx}{f(x)} = \Sigma_n{}^1 \frac{1}{f'(a_r)} \log\left(\frac{x-a_r}{x+a_r}\right),$$

where $\quad f(x) = (x^2-a_1^2)(x^2-a_2^2)\ldots(x^2-a_n^2),$

and the summation refers to the letter r.

28. We now consider the case in which $f(x)$ has a factor of the form $(x-b)^p$. Putting

$$f(x) = (x-b)^p f_1(x),$$

we may assume

$$\frac{\phi(x)}{(x-b)^p f_1(x)} = \frac{B_1}{(x-b)^p} + \frac{B_2}{(x-b)^{p-1}} + \ldots + \frac{B_p}{x-b} + \frac{\phi_1(x)}{f_1(x)}, \qquad (6)$$

where $\phi_1(x)$ is an integral expression of the $(n-p-1)^{th}$ degree in x, and $B_1, B_2, \ldots B_p$ are constants. This assumption is legitimate; for if we multiply both sides of (6) by $(x-b)^p f_1(x)$, we obtain

$$\phi(x) = \{B_1 + B_2(x-b) + \ldots + B_p(x-b)^{p-1}\} f_1(x) + \phi_1(x)(x-b)^p; \quad (7)$$

and equating then like powers of x in this identity, we get as many equations as there are indeterminate quantities.

To find the actual values of B_1, B_2, &c., let $x = b$ in (7). We get thus

$$\phi(b) = B_1 f_1(b),$$

which determines B_1. Again, differentiating (7) with regard to x, and then putting $x = b$, we find

$$\phi'(b) = B_1 f_1'(b) + B_2 f_1(b).$$

Continuing this process, we obtain

$$\phi''(b) = B_1 f_1''(b) + 2B_2 f_1'(b) + 2B_3 f_1(b),$$

$$\phi'''(b) = B_1 f_1'''(b) + 3B_2 f_1''(b) + 6B_3 f_1'(b) + 6B_4 f_1(b),$$

and so on. These equations evidently enable us to determine B_1, B_2, ... B_n in succession. We might, however, write down at once the value of B_r as follows:—Multiply both sides of (6) by $(x-b)^p$, differentiate r times with respect to x, and then put $x = b$. We thus find

$$B_r = \frac{1}{\underline{|r}} \frac{d^r}{dx^r} \left\{\frac{\phi(x)}{f_1(x)}\right\}, \tag{8}$$

where b is to be put for x after differentiation.

Integrating now both sides of the equation (6), we obtain

$$\int \frac{\phi(x)\,dx}{(x-b)^p f_1(x)} = \int \frac{\phi_1(x)\,dx}{f_1(x)} + B_p \log(x-b) - \frac{B_{p-1}}{x-b}$$

$$- \frac{1}{2}\frac{B_{p-2}}{(x-b)^2} \cdots - \frac{B_1}{(p-1)(x-b)^{p-1}}. \tag{9}$$

If $f(x)$ have any other factor of the form $(x-c)^q$, we must evidently put $f_1(x) = (x-c)^q f_2(x)$, and repeat the process just given for the integral

$$\int \frac{\phi_1(x)}{f_1(x)}\,dx.$$

29. We may notice here another method of treating the integral in the case considered in the preceding Article. Let $x = b + \frac{1}{y}$, then $dx = -\frac{dy}{y^2}$, and $\frac{dx}{(x-b)^p} = -y^{p-2}\,dy$.

Also, we have

$$\phi(x) = \phi\left(b + \frac{1}{y}\right) = \frac{1}{y^{n-1}}\psi(y),$$

$$f_1(x) = f_1\left(b + \frac{1}{y}\right) = \frac{1}{y^{n-p}}\psi_1(y),$$

where $\psi(y)$, $\psi_1(y)$ are integral expressions of the degrees $n-1$ and $n-p$, respectively. We get thus

$$\int \frac{\phi(x)\,dx}{(x-b)^p f_1(x)} = -\int \frac{\psi(y)\,dy}{y\psi_1(y)}.$$

Now since the degree of $\psi(y)$ exceeds that of $y\psi_1(y)$ by p, we can evidently evaluate the latter integral in the manner already mentioned in Art. 25. In fact, by the transformation just made use of, the given integral is resolved into the sum of two integrals, namely, that of a rational integral expression, and another of the same form as the given one, from which the multiple factor has disappeared. This method will be found useful when $f(x)$ has but one factor of the form $(x-b)^p$.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{(x-1)^2(x-2)} = \frac{1}{x-1} + \log\left(\frac{x-2}{x-1}\right).$

2. $\displaystyle\int \frac{x\,dx}{(x-1)^2(x-2)} = \frac{1}{x-1} + 2\log\left(\frac{x-2}{x-1}\right).$

3. $\displaystyle\int \frac{dx}{x^2(a-x)} = \frac{1}{ax} - \frac{1}{a^2}\log\left(\frac{x}{a-x}\right).$

4. $\displaystyle\int \frac{dx}{(x-a)^2(x-b)^2} = -\frac{1}{(a-b)^2}\frac{(2x-a-b)}{(x-a)(x-b)} + \frac{2}{(a-b)^3}\log\left(\frac{x-b}{x-a}\right).$

5. $\displaystyle\int \frac{dx}{x(x^2-a^2)^2} = \frac{1}{2}\int \frac{dz}{z(z-a^2)^2}$, putting $x^2 = z$.

We thus find

$$\int \frac{dx}{x(x^2-a^2)^2} = \frac{1}{2a^2}\,\frac{1}{x^2-a^2} + \frac{1}{2a^4}\log\left(1-\frac{a^2}{x^2}\right).$$

6. $\displaystyle\int \frac{dx}{(x^2-a^2)^2} = \frac{1}{4a^3}\log\left(\frac{x+a}{x-a}\right) + \frac{1}{2a^2}\,\frac{x}{x^2-a^2}.$

7. $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^2}.$

This integral may be made to depend upon the preceding one as follows. We have

$$\int \frac{dx}{x^2(x^2-a^2)^2} = \int \frac{dx}{a^4}\left\{\frac{x}{x^2-a^2}-\frac{1}{x}\right\}^2$$

$$= \int \frac{dx}{a^4}\left\{\frac{x^2}{(x^2-a^2)^2}-\frac{2}{x^2-a^2}+\frac{1}{x^2}\right\}$$

$$= \int \frac{dx}{a^4}\left\{\frac{a^2}{(x^2-a^2)^2}-\frac{1}{x^2-a^2}+\frac{1}{x^2}\right\}$$

$$= \frac{1}{2a^5}\log\left(\frac{x+a}{x-a}\right) - \frac{1}{a^4x} + \frac{1}{a^2}\int \frac{dx}{(x^2-a^2)^2}.$$

8. $\displaystyle\int \frac{dx}{(x-a)^3 x} = \frac{2x-3a}{2a^2(x-a)^2} + \frac{1}{a^3}\log\left(1-\frac{a}{x}\right).$

9. $\displaystyle\int \left\{\frac{\phi(x)}{f(x)}\right\}^2 dx,$

where $\phi(x) = ax^2+bx+c$, $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$.

We have $$\frac{\phi(x)}{f(x)} = \frac{l}{x-\alpha}+\frac{m}{x-\beta}+\frac{n}{x-\gamma},$$

where $$l = \frac{\phi(\alpha)}{f'(\alpha)},\quad m = \frac{\phi(\beta)}{f'(\beta)},\quad n = \frac{\phi(\gamma)}{f'(\gamma)}.$$

Hence, $$\left\{\frac{\phi(x)}{f(x)}\right\}^2 = \frac{l^2}{(x-\alpha)^2}+\frac{m^2}{(x-\beta)^2}+\frac{n^2}{(x-\gamma)^2}+\frac{2lm}{(x-\alpha)(x-\beta)}$$

$$+\frac{2mn}{(x-\beta)(x-\gamma)}+\frac{2nl}{(x-\alpha)(x-\gamma)},$$

and the integral sought is equal to

$$-\left\{\frac{l^2}{x-\alpha}+\frac{m^2}{x-\beta}+\frac{n^2}{x-\gamma}\right\}+\frac{2lm}{\alpha-\beta}\log\left(\frac{x-\alpha}{x-\beta}\right)+\frac{2mn}{\beta-\gamma}\log\left(\frac{x-\beta}{x-\gamma}\right)$$
$$+\frac{2nl}{\gamma-\alpha}\log\left(\frac{x-\gamma}{x-\alpha}\right).$$

10. $$\int\frac{dx}{(x-\alpha)^2(x-\beta)(x-\gamma)}=\frac{1}{(\alpha-\beta)(\alpha-\gamma)}\frac{1}{x-\alpha}$$
$$+\frac{(2\alpha-\beta-\gamma)}{(\alpha-\beta)^2(\alpha-\gamma)^2}\log(x-\alpha)+\frac{\log(x-\beta)}{(\alpha-\beta)^2(\beta-\gamma)}-\frac{\log(x-\gamma)}{(\alpha-\gamma)^2(\beta-\gamma)}.$$

30. We consider now the case in which $f(x)$ has a factor of the form $x^2-2ax+a^2+\beta^2$, that is, when it has imaginary roots. The results already obtained for real roots will, of course, still hold; but as each separate term involves the imaginary, it is necessary to show that, if we take together the terms arising from conjugate imaginary roots, the sum so obtained will be real.

Let $a\pm i\beta$ be two conjugate imaginary roots of $f(x)$, then the corresponding part in the decomposition of $\phi(x)/f(x)$ into partial fractions is, from (4),

$$\frac{\phi(a+i\beta)}{f'(a+i\beta)}\,\frac{1}{x-a-i\beta}+\frac{\phi(a-i\beta)}{f'(a-i\beta)}\,\frac{1}{x-a+i\beta},$$

or
$$\frac{L(x-a)-M}{(x-a)^2+\beta^2}, \tag{10}$$

if we put

$$\frac{\phi(a+i\beta)}{f'(a+i\beta)}+\frac{\phi(a-i\beta)}{f'(a-i\beta)}=L,$$

$$\frac{\phi(a+i\beta)}{f'(a+i\beta)}-\frac{\phi(a-i\beta)}{f'(a-i\beta)}=iM.$$

But it is easy to see that L, M, and therefore (10) are real. The corresponding expression in the integral

$$\int \frac{\phi(x)}{f(x)}\,dx$$

is then $\qquad \frac{1}{2} L \log\{(x-a)^2+\beta^2\} - \frac{M}{\beta}\tan^{-1}\left(\frac{x-a}{\beta}\right). \qquad (11)$

31. Practically, however, the method just given is not, in most cases, the best for obtaining the values of L, M. If we put $f(x) = (x^2 - 2ax + a^2 + \beta^2)\,\chi(x)$, we may write

$$\frac{\phi(x)}{(x^2-2ax+a^2+\beta^2)\,\chi(x)} = \frac{L(x-a)-M}{x^2-2ax+a^2+\beta^2} + \frac{R}{S}, \qquad (12)$$

where R/S is a fraction arising from the factors of $\chi(x)$. We have then

$$\phi(x) = \{L(x-a)-M\}\,\chi(x) + \frac{R}{S}(x^2-2ax+a^2+\beta^2)\,\chi(x),$$

and, therefore,

$$\phi(x) = \{L(x-a)-M\}\,\chi(x), \qquad (13)$$

for both the values $a \pm i\beta$ of x. Substituting now $2ax - a^2 - \beta^2$ for x^2 in (13), and repeating this operation, we shall finally get a simple equation in x. Equating then the terms on both sides of this equation, we get the values of L, M.

EXAMPLES.

1. $\int \frac{dx}{1+x^3}$.

Let $$\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1-x+x^2};$$

then, from (3), $A = \frac{1}{3}$. Clearing of fractions, we get

$$= 1 - x + x^2 + 3(1+x)(Bx+C),$$

whence, from the coefficient of x^2 and the absolute term we find $B = -\frac{1}{3}$, $C = \frac{2}{3}$. We have thus—

$$\int \frac{dx}{1+x^3} = \frac{1}{6} \log \left\{\frac{(1+x)^2}{1-x+x^2}\right\} + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}}\right).$$

2. $$\int \frac{x\,dx}{1+x^3} = \frac{1}{6} \log \left\{\frac{1-x+x^2}{(1+x)^2}\right\} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}.$$

3. $$\int \frac{(mx+n)}{x^4-1}\,dx = \frac{m}{4} \log \left(\frac{x^2-1}{x^2+1}\right) + \frac{n}{4} \log \left(\frac{x-1}{x+1}\right) + \frac{n}{2} \tan^{-1} x.$$

4. $$\int \frac{dx}{(x-a)(1+x^2)} = \frac{1}{1+a^2} \log \left(\frac{x-a}{\sqrt{(1+x^2)}}\right) - \frac{a}{1+a^2} \tan^{-1} x.$$

5. $$\int \frac{dx}{(x-a)^2(1+x^2} = \frac{1}{(1+a^2)(x-a)} + \frac{2a}{(1+a^2)^2} \log \left(\frac{x-a}{\sqrt{(1+x^2)}}\right)$$
$$- \frac{a^2-1}{(a^2+1)^2} \tan^{-1} x.$$

This integral may also be found from the preceding by differentiation with regard to a.

6. $$\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2 x} + \frac{1}{a^3} \cot^{-1} \left(\frac{x}{a}\right)$$

7. $$\int \frac{dx}{1-x^6} = \frac{1}{12} \log \left\{\frac{(1+x)^2}{(1-x)^2} \frac{(1+x+x^2)}{(1-x+x^2)}\right\} + \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x\sqrt{3}}{1-x^2}\right).$$

8. $$\int \frac{dx}{x^4-4x+3}.$$

We have $$x^4 - 4x + 3 = (x-1)^2(x^2+2x+3).$$

Hence, assuming

$$\frac{1}{x^4-4x+3} = \frac{\alpha}{(x-1)^2} + \frac{\beta}{x-1} + \frac{\gamma x + \delta}{x^2+2x+3},$$

we have $$1 = \alpha(x^2+2x+3) + \beta(x-1)(x^2+2x+3) + (\gamma x + \delta)(x-1)^2.$$

Putting then $x = 1$, we get $\alpha = \frac{1}{6}$; and taking $x^2 + 2x + 3 = 0$, we have

$$1 = (\gamma x + \delta)(1 - 2x - 2x - 3),$$

or $$4x(\gamma x+\delta)+2(\gamma x+\delta)+1=0;$$

hence $$-4\gamma(2x+3)+4\delta x+2(\gamma x+\delta)+1=0,$$

from which we find $\gamma=\frac{1}{9}$, $\delta=\frac{1}{6}$. From the coefficient of x^3 we obtain then

$$\beta=-\gamma=-\tfrac{1}{9},$$

so that finally we get

$$\int\frac{dx}{x^4-4x+3}=-\frac{1}{6}\frac{1}{x-1}-\frac{1}{9}\log(x-1)+\frac{1}{18}\log(x^2+2x+3)$$
$$+\frac{1}{18\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right).$$

The integral may, however, be obtained more easily by putting $x=1+z^{-1}$.

9. $$\int\frac{x\,dx}{x^4-4x+3}=-\frac{1}{6}\frac{1}{x-1}+\frac{1}{18}\log(x-1)-\frac{1}{36}\log(x^2+2x+3).$$
$$-\frac{5}{18\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right).$$

10. $$\int\frac{dx}{x^4+4}=\tfrac{1}{8}\int\left\{\frac{2-x}{x^2-2x+2}+\frac{2+x}{x^2+2x+2}\right\}dx$$
$$=\tfrac{1}{8}\{\tan^{-1}(x-1)+\log\sqrt{(x^2-2x+2)}$$
$$+\tan^{-1}(x+1)+\log\sqrt{(x^2+2x+2)}\}$$
$$=\tfrac{1}{8}\tan^{-1}\frac{2x}{2-x^2}+\tfrac{1}{16}\log\left\{\frac{x^2+2x+2}{x^2-2x+2}\right\}.$$

11. $$\int\frac{x^3\,dx}{(x-1)^2(x^2-2x+2)}=-\frac{1}{x-1}+\log\left\{\frac{(x-1)^3}{x^2-2x+2}\right\}+2\tan^{-1}(x-1).$$

12. $$\int\frac{dx}{x^5-10x^2+15x-6}=\int\frac{dx}{(x-1)^3(x^2+3x+6)}$$
$$=-\int\frac{z^3\,dz}{10z^2+5z+1},\ \text{where } z=\frac{1}{x-1}.$$

13. $$\int\frac{(x+3)\,dx}{4x^3+9x^2+18x+17}=\int\frac{8(x+3)\,dx}{(3x+1)^3+5(x+3)^3}$$
$$=4\int\frac{dz}{z^3+5},\ \text{where } z=\frac{3x+1}{x+3}.$$

14. $$\int \frac{dx}{(x^3-1)(x^2+1)} = \tfrac{1}{6}\int \left\{\frac{-1}{x-1} - \frac{3(x-1)}{x^2+1} + \frac{2(2x+1)}{x^2+x+1}\right\} dx$$

$$= \tfrac{1}{6}\log\left\{\frac{(x-1)(x^2+1)^{\frac{3}{2}}}{(x^2+x+1)^2}\right\} - \tfrac{1}{2}\tan^{-1} x.$$

15. $$\int \frac{x^2\,dx}{(x^2+1)(x^2+4)(x^2+9)} = \tfrac{1}{24}\tan^{-1} x - \tfrac{2}{15}\tan^{-1}\left(\frac{x}{2}\right) + \tfrac{3}{40}\tan^{-1}\left(\frac{x}{3}\right).$$

16. $$\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\int\left(\frac{a^2}{x^2+a^2} - \frac{b^2}{x^2+b^2}\right) dx$$

$$= \frac{1}{a^2-b^2}\left(a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right).$$

32. The case when $f(x)$ has a factor of the form

$$(x^2 - 2ax + a^2 + \beta^2)^q$$

can be made to depend upon Art. 28. Putting

$$f(x) = (x^2 - 2ax + a^2 + \beta^2)^q\,\chi(x) = (x - a - i\beta)^q\,(x - a + i\beta)^q\,\chi(x),$$

$$\frac{\phi(x)}{(x - a + i\beta)^q\,(x - a + i\beta)^q\,\chi(x)}$$

can be expressed, as we have already shown, as follows

$$\frac{B_1}{(x-a-i\beta)^q} + \frac{B_2}{(x-a-i\beta)^{q-1}} + \ldots + \frac{B_q}{x-a-i\beta}$$

$$+ \frac{C_1}{(x-a+i\beta)^q} + \frac{C_2}{(x-a+i\beta)^{q-1}} + \ldots + \frac{C_q}{x-a+i\beta}$$

$$+ \frac{\phi_1(x)}{\chi(x)}. \qquad (14)$$

Now it is easy to see that B_r and C_r must be conjugate imaginaries. Hence, putting

$$B_r = L_r + iM_r, \quad C_r = L_r - iM_r,$$

and taking together the terms

$$\frac{L_r + iM_r}{(x-a-i\beta)^r} + \frac{L_r - iM_r}{(x-a+i\beta)^r}, \tag{15}$$

the sum is real; for, letting

$$x - a = \rho\cos\theta, \quad \beta = \rho\sin\theta,$$

(15) becomes $\quad \dfrac{1}{\rho^r}(L_r + iM_r)(\cos r\theta + i\sin r\theta)$

$$+ \frac{1}{\rho^r}(L_r - iM_r)(\cos r\theta - i\sin r\theta) = \frac{2}{\rho^n}(L_r\cos r\theta - M_r\sin r\theta).$$

Multiplying (15) then by dx, and integrating, we get

$$-\frac{1}{r-1}\left\{\frac{L_r + iM_r}{(x-a-i\beta)^{r-1}} + \frac{L_r - iM_r}{(x-a+i\beta)^{r-1}}\right\}$$

$$= \frac{-2}{(r-1)\rho^{r-1}}\{L_r\cos(r-1)\theta - M_r\sin(r-1)\theta\}, \tag{16}$$

except in the case in which $r = 1$, when we have

$$L_1\log\{(x-a)^2 + \beta^2\} + 2M_1\tan^{-1}\left(\frac{\beta}{x-a}\right). \tag{17}$$

33. We might also consider the integral in the preceding Article without making any use of the imaginary. We may assume

$$\frac{\phi(x)}{(x^2 - 2ax + a^2 + \beta^2)^q\,\chi(x)} = \frac{A_1x + B_1}{(x^2 - 2ax + a^2 + \beta^2)^q}$$

$$+ \frac{A_2x + B_2}{(x^2 - 2ax + a^2 + \beta^2)^{q-1}} + \ldots + \frac{A_qx + B_q}{x^2 - 2ax + a^2 + \beta^2} + \frac{\phi_1(x)}{\chi(x)}; \tag{18}$$

for, if we clear of fractions, and then equate the coefficients of the different powers of x, we get the proper number of equations to determine L_1, M_1, L_2, &c. The latter quantities may,

in many cases, be most easily determined directly from these $2q$ linear equations. The given integral is thus made to depend upon a series of integrals of the form

$$\int \frac{(A_r x + B_r)\,dx}{(x^2 - 2ax + a^2 + \beta^2)^r},$$

which is equal to

$$\frac{-A_r}{2(r-1)(x^2 - 2ax + a^2 + \beta^2)^{r-1}} + (A_r a + B_r) \int \frac{dx}{(x^2 - 2ax + a^2 + \beta^2)^r}.$$

Now, putting $x = a + y$, we have

$$\int \frac{dx}{(x^2 - 2ax + a^2 + \beta^2)^r} = \int \frac{dy}{(y^2 + \beta^2)^r};$$

but $$\int \frac{dy}{(y^2 + \beta^2)^r} = \int \frac{y^{-3}}{(1 + \beta^2 y^{-2})^r}\, y^{-2r+3}\,dy,$$

which, being integrated by parts, becomes

$$\frac{y^{-2r+3}}{2(r-1)\beta^2 (1 + \beta^2 y^{-2})^{r-1}} + \frac{2r-3}{(2r-2)\beta^2} \int (1 + \beta^2 y^{-2})^{-r+1} y^{-2r+2}\,dy$$

$$= \frac{y}{2(r-1)\beta^2 (y^2 + \beta^2)^{r-1}} + \frac{2r-3}{2(r-1)\beta^2} \int \frac{dy}{(y^2 + \beta^2)^{r-1}}, \tag{19}$$

from which result, by changing r successively into $r - 1$, $r - 2$, &c., we shall make

$$\int \frac{dy}{(y^2 + \beta^2)^r} \text{ depend upon } \int \frac{dy}{y^2 + \beta^2} = \frac{1}{\beta} \tan^{-1}\left(\frac{y}{\beta}\right).$$

The integration of $dy / (y^2 + \beta^2)^r$, in the manner just given, furnishes an example of the method (4) of Art. 5, namely, that of integration by successive reduction.

EXAMPLES.

1. $$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2a^2}\,\frac{x}{x^2+a^2}.$$

2. $$\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^5}\left\{\tfrac{3}{2}\tan^{-1}\left(\frac{x}{a}\right) + \tfrac{1}{2}\frac{ax}{a^2+x^2} + \frac{a}{x}\right\}.$$

3. $$\int \frac{dx}{(x^2-2x\cos\alpha+1)^2} = \frac{1}{2\sin^3\alpha}\left\{\tan^{-1}\left(\frac{x-\cos\alpha}{\sin\alpha}\right) + \frac{\sin\alpha\,(x-\cos\alpha)}{x^2-2x\cos\alpha+1}\right\}.$$

4. $$\int \frac{x\,dx}{(x^2-2x\cos\alpha+1)^2} = \frac{\cos\alpha}{2\sin^3\alpha}\tan^{-1}\left(\frac{x-\cos\alpha}{\sin\alpha}\right) - \frac{1}{2\sin^2\alpha}\,\frac{(x-\cos\alpha+\sin^2\alpha)}{x^2-2x\cos\alpha+1}.$$

5. $$\int \frac{dx}{(1+x^3)^2} = \frac{1}{9}\log\left(\frac{(1+x)^2}{x^2-x+1}\right) + \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) - \frac{1}{9}\,\frac{1}{1+x} + \frac{1}{9}\,\frac{(1+x)}{x^2-x+1}.$$

6. $$\int \frac{dx}{x(1+x^3)^2} = \tfrac{1}{3}\int \frac{dz}{z(1+z)^2}, \text{ if } x^3 = z.$$

Hence $$\int \frac{dx}{x(1+x^3)^2} = \tfrac{1}{3}\log\left(\frac{x^3}{1+x^3}\right) + \tfrac{1}{3}\,\frac{1}{1+x^3}.$$

34. We have given already, in Art. 21, a method of evaluating the integral $R\,dx/PQ$, or rather

$$\int \frac{R(x\,dy-y\,dx)}{PQ}, \tag{20}$$

where P, Q, R are homogeneous quadratic expressions in x, y. This method may be extended to the more general integral

$$\int \frac{U_{n-2}(x\,dy-y\,dx)}{U_n},$$

where U_n and U_{n-2} are homogeneous integral expressions in x, y of the degrees n and $n-2$, respectively, and n is an even number.

First, let $n = 6$, then we know that we may in general assume $U_6 = P_1 P_2 P_3$, where P_1, P_2, P_3, are real quadratic expressions. Now let J_{23}, J_{31}, J_{12} denote the three Jacobian quadratics of P_2, P_3; P_3, P_1; P_1, P_2, respectively; then suppose we assume

$$U_4 = lP_2P_3 + mP_3P_1 + nP_1P_2 + \lambda J_{21}P_1 + \mu J_{31}P_2 + \nu J_{12}P_3. \quad (21)$$

But putting for P_1, P_2, P_3 the values

$$\frac{1}{2}\left(x\frac{dP_1}{dx} + y\frac{dP_1}{dy}\right), \text{ \&c.},$$

we see that

$$J_1P_1 + J_2P_2 + J_3P_3 = 0, \qquad (22)$$

identically. Equating then the coefficients on both sides, we get a sufficient number of linear equations to determine l, m, n and the differences of λ, μ, ν, as we see from (22). We thus get, making use of a result given in Art. 21,

$$\int\frac{U_4(x\,dy - y\,dx)}{P_1P_2P_3} = l\int\frac{(x\,dy - y\,dx)}{P_1} + m\int\frac{(x\,dy - y\,dx)}{P_2}$$
$$+ n\int\frac{(x\,dy - y\,dx)}{P_3} + 2\{(\mu - \nu)\log P_1 + (\nu - \lambda)\log P_2$$
$$+ (\lambda - \mu)\log P_3\}. \qquad (23)$$

Each of the integrals

$$\int\frac{x\,dy - y\,dx}{P_1}, \text{ \&c.},$$

can then be evaluated by (39) or (40), in Chap. I.

In the same way we can integrate

$$\frac{U_6}{U_8}(x\,dy - y\,dx).$$

Putting $U_8 = P_1P_2P_3P_4$, and assuming

$$U_6 = lP_2P_3P_4 + mP_1P_3P_4 + nP_1P_2P_4 + pP_1P_2P_3$$
$$+ \lambda J_{12}P_3P_4 + \mu J_{23}P_4P_1 + \nu J_{34}P_1P_2, \qquad (24)$$

we can determine the seven quantities $l, m, n, p, \lambda, \mu, \nu$. The given integral is thus made to depend upon the four integrals of the form

$$\int \frac{x\,dy - y\,dx}{P_r},$$

besides the expression

$$2\left\{\lambda \log\left(\frac{P_1}{P_2}\right) + \mu \log\left(\frac{P_2}{P_3}\right) + \nu \log\left(\frac{P_3}{P_4}\right)\right\}.$$

It may be observed that the case, when n is odd, can be made to depend upon the preceding by multiplying both U_n and U_{n-2} by an arbitrary linear factor $lx + my$.

35. In some cases in which the roots of $f(x)$ are known, the partial fractions arising from the decomposition of $\phi(x)/f(x)$ take a simple form, so that we may at once write down the integral as the sum of a series of known terms. For instance, let us consider the integral

$$\int \frac{x^m\,dx}{x^n - 1}, \qquad (25)$$

where m is less than n.

Now if a is an imaginary root of $x^n - 1 = 0$, we know that a^{-1} is also; in fact these two roots are, by trigonometrical considerations, of the form

$$\cos\frac{2r\pi}{n} \pm i\sin\frac{2r\pi}{n},$$

where r is some integer. But the partial fractions corresponding to these roots are, from (3),

$$\frac{a^m}{na^{n-1}(x-a)}, \text{ or } \frac{a^{m+1}}{n(x-a)}, \text{ and } \frac{a^{-(m+1)}}{n(x-a^{-1})}.$$

Hence, adding these terms together, we get

$$\frac{1}{n}\frac{x(a^{m+1}+a^{-m-1})-(a^m+a^{-m})}{x^2-(a+a^{-1})x+1} = \frac{2}{n}\frac{x\cos(m+1)\phi - \cos m\phi}{x^2-2x\cos\phi+1},$$

if we put $2r\pi/n = \phi$. Writing then this expression in the form

$$\frac{2}{n}\frac{\cos(m+1)\phi\,(x-\cos\phi)}{x^2-2x\cos\phi+1} - \frac{2}{n}\frac{\sin(m+1)\phi\sin\phi}{(x-\cos\phi)^2+\sin^2\phi},$$

multiplying by dx, and integrating, we obtain

$$\frac{1}{n}\cos(m+1)\phi\log(x^2-2x\cos\phi+1)$$
$$-\frac{2}{n}\sin(m+1)\phi\tan^{-1}\left(\frac{x-\cos\phi}{\sin\phi}\right). \qquad (26)$$

We have now to consider two cases, according as n is even or odd.

First, let $n = 2p$, then $x^{2p} - 1$ has $x-1$ and $x+1$ as factors, and we easily find

$$\int \frac{x^m\,dx}{x^{2p}-1} = \frac{1}{2p}\log(x-1) + \frac{(-1)^{m+1}}{2p}\log(x+1)$$

$$+\frac{1}{2p}\Sigma_1^{p-1}\cos\left(\frac{(m+1)r\pi}{p}\right)\log\left(1-2x\cos\frac{2\pi}{p}+x^2\right)$$

$$-\frac{1}{p}\Sigma_1^{p-1}\sin\left(\frac{(m+1)\,r\pi}{p}\right)\tan^{-1}\left(\frac{x-\cos\frac{r\pi}{p}}{\sin\frac{r\pi}{p}}\right). \quad (27)$$

Secondly, let $n = 2p+1$, then we find

$$\int \frac{x^m\,dx}{x^{2p+1}-1} = \frac{\log(x-1)}{2p+1} + \frac{1}{2p+1}\Sigma_1^{p}\cos\left\{\frac{(m+1)\,2r\pi}{2p+1}\right\}$$

$$\times\log\left(1-2x\cos\frac{2r\pi}{2p+1}+x^2\right)$$

$$+\frac{2}{2p+1}\Sigma_1^{p}\sin\left(\frac{(m+1)\,2r\pi}{2p+1}\right)\tan^{-1}\left(\frac{x-\cos\frac{2r\pi}{2p+1}}{\sin\frac{2r\pi}{2p+1}}\right).$$

In both of these results the summation is taken with regard to the letter r between the limits indicated in each case.

Examples.

1. $\displaystyle\int \frac{x\,dx}{x^2-2x-2} = \frac{1}{2}\log(x^2-2x-2) + \frac{1}{2\sqrt{3}}\log\left(\frac{x+1-\sqrt{3}}{x+1+\sqrt{3}}\right).$

2. $\displaystyle\int \frac{dx}{x^3-13x+12} = \tfrac{1}{70}\log\left\{\frac{(x-3)^5(x+4)^2}{(x-1)^7}\right\}.$

3. $\displaystyle\int \frac{dx}{x^2(x-1)(x-2)} = \frac{1}{2x} + \frac{1}{4}\log\left\{\frac{x-2}{x^3(x-1)}\right\}.$

4. $\displaystyle\int \frac{dx}{(x-n)(x^2-1)} = \frac{1}{2(n^2-1)} \log \left\{ \frac{(x-n)^2}{x^2-1} \left(\frac{x+1}{x-1} \right)^n \right\}.$

5. $\displaystyle\int \frac{(x^2-a^2)\,dx}{x^4+6a^2x^2+a^4} = \frac{1}{2a} \tan^{-1} \left(\frac{x^2+a^2}{2ax} \right).$

6. $\displaystyle\int \frac{dx}{(1-x^4)^2} = \tfrac{1}{4} \frac{x}{1-x^4} + \tfrac{3}{8} \tan^{-1} x + \tfrac{3}{16} \log \tfrac{3}{8} \tan^{-1} x + \tfrac{3}{16} \log \left(\frac{1+x}{1-x} \right).$

7. $\displaystyle\int \frac{dx}{x(1+x+x^2+x^3)} = \tfrac{1}{4} \log \left\{ \frac{x^4}{(1+x)^2(1+x^2)} \right\} + \tfrac{1}{2} \tan^{-1} x.$

8. $\displaystyle\int \frac{(mx+n)\,dx}{(x+1)^2(x+3)} = \frac{1}{2} \frac{m-n}{x+1} - \frac{(3m-n)}{4} \log \cdot \left(\frac{x+3}{x+1} \right).$

9. $\displaystyle\int \frac{x^2\,dx}{x^4-2a^2x^2\cos 2\alpha+a^4} = \frac{1}{8a\cos\alpha} \log \left\{ \frac{x^2-2ax\cos\alpha+a^2}{x^2+2ax\cos\alpha+a^2} \right\}$

$\displaystyle+ \frac{1}{4a\sin\alpha} \tan^{-1} \left(\frac{2ax\sin\alpha}{a^2-x^2} \right).$

10. $\displaystyle\int \frac{dx}{x^4-2a^2x^2\cos 2\alpha+a^4} = -\frac{1}{8a^3\cos\alpha} \log \left(\frac{x^2-2ax\cos\alpha+a^2}{x^2+2ax\cos\alpha+a^2} \right)$

$\displaystyle+ \frac{1}{4a^3\sin\alpha} \tan^{-1} \left(\frac{2ax\sin\alpha}{a^2-x^2} \right).$

11. $\displaystyle\int \frac{(2x+3)\,dx}{x(x^2+x-2)} = \tfrac{1}{6} \log \left\{ \frac{(x-1)^{10}}{x^9(x+2)} \right\}.$

12. $\displaystyle\int \frac{dx}{x^4+5x^2+4} = \tfrac{1}{6} \tan^{-1} \left(\frac{x^3+3x}{2-4x^2} \right).$

13. $\displaystyle\int \frac{x^2\,dx}{(1+x^2)(1+4x^2)} = \tfrac{1}{3} \tan^{-1} \left(\frac{2x^3}{1+3x^2} \right).$

14. $\displaystyle\int \frac{dx}{(1+x+x^2)^2} = \frac{2x+1}{3(1+x+x^2)} + \frac{4}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right).$

15. $\displaystyle\int \frac{x\,dx}{(1+x+x^2)^2} = -\frac{(x+2)}{3(1+x+x^2)} \overset{-}{\cancel{+}} \frac{2}{3\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right).$

16. $\displaystyle\int \frac{x^5\,dx}{(1-x^2)^2} = \frac{1}{2(1-x^2)} + \frac{x^2}{2} + \log(1-x^2).$

17. $$\int \frac{(3x-2)\,dx}{(x-2)^2(x+2)} = \tfrac{1}{2}\log\left(\frac{x-2}{x+2}\right) - \frac{1}{x-2}.$$

18. $$\int \frac{d\theta}{\sin 2\theta - n\sin\theta} = \frac{1}{n^2-4}\left\{2\log\left(\frac{2\cos\theta - n}{\sin\theta}\right) - n\log\tan\frac{\theta}{2}\right\}.$$

19. $$\int \frac{d\theta}{\cos\theta\,(1+m^2\sin^2\theta)} = \frac{1}{1+m^2}\log(\tan\theta+\sec\theta) + \frac{m}{1+m^2}\tan^{-1}(m\sin\theta).$$

20. $$\int \frac{x^2\,dx}{(x^2+1)(x^2+2)(x^2+4)} = \frac{1}{3}\tan^{-1}\frac{3x}{2-x^2} + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right).$$

21. $$\int \frac{x^2\,dx}{(x^2+1)(x^2+4)(x^2+16)} = \tfrac{1}{45}\tan^{-1}\left(\frac{5x}{4-x^2}\right) + \tfrac{1}{18}\tan^{-1}\frac{x}{2}.$$

22. $$\int \frac{(x^2-x+1)\,dx}{(1+x)(1+x^2)} = \tfrac{1}{4}\log\left\{\frac{(x+1)^6}{1+x^2}\right\} - \tfrac{1}{2}\tan^{-1}x.$$

23. $$\int \frac{(ax+b)^{m+n-2}\,dx}{(x-\alpha)^m(x-\beta)^n} = \frac{1}{(\alpha-\beta)^{m+n-1}}\int (p-qz)^{m+n-2}\frac{dz}{z^m},$$

where $$x = \frac{\alpha-\beta z}{1-z},\quad a+b\alpha = p,\quad a+b\beta = q.$$

The latter integral may be evaluated by expanding $(p-qz)^{m+n-2}$ in a series of a limited number of terms.

24. $$\int \frac{dx}{(x-\alpha)^2(x-\beta)^3} = \frac{(x-\alpha)^2 - 5(x-\alpha)(x-\beta) - 2(x-\beta)^2}{2(\alpha-\beta)^3(x-\alpha)(x-\beta)^2} - \frac{3}{(\alpha-\beta)^4}\log\left(\frac{x-\alpha}{x-\beta}\right).$$

25. $$\int \frac{dx}{x(1+x^n)^m} = -\frac{1}{n}\int (z-1)^{m-1}\frac{dz}{z^m},$$

where $$x^n(z-1) = 1.$$

26. $$\int \frac{(ax^2+2bxy+cy^2)(x\,dy-y\,dx)}{x^4+y^4-2x^2y^2\cos 2\theta}$$
$$= l\tan^{-1}\left\{\frac{x+y}{x-y}\tan\frac{\theta}{2}\right\} + m\tan^{-1}\left\{\left(\frac{x+y}{x-y}\right)\cot\frac{\theta}{2}\right\}$$
$$+ n\log\left\{\frac{x^2+y^2+2xy\cos\theta}{x^2+y^2-2xy\cos\theta}\right\},$$

where
$$4l = \frac{a + c + 2h \sec\theta}{\sin\theta},$$

$$4m = \frac{a + c - 2h \sec\theta}{\sin\theta},$$

$$4n = (a - c)\sec\theta.$$

27. Show that

$$\int \frac{dx}{1+x^{2n}} = -\frac{1}{2n}\,\Sigma_n^1 \cos\frac{(2r-1)\pi}{2n} \log\left(1 - 2x\cos\frac{(2r-1)\pi}{2n} + x^2\right)$$

$$+\frac{1}{n}\,\Sigma_n^1 \sin\frac{(2r-1)\pi}{2n} \tan^{-1}\left\{\frac{x - \cos\dfrac{(2r-1)\pi}{2n}}{\sin\dfrac{(2r-1)\pi}{2n}}\right\},$$

where the summation refers to the letter r.

28. Show that

$$\int \frac{(x^{m-1} + x^{n-m-1})\,dx}{1+x^n} = \frac{4}{n}\,\Sigma \sin\frac{(2r-1)\,m\pi}{n} \tan^{-1}\left(\frac{x - \cos\dfrac{(2r-1)\,m\pi}{n}}{\sin\dfrac{(2r-1)\,m\pi}{n}}\right),$$

where r is taken between 1 and $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$, according as n is even or odd.

29. Show that the general term of the integral

$$\int \frac{x^m\,dx}{x^{2n} - 2x^n\cos\theta + 1}$$

is

$$\frac{1}{n\sin\theta}\cos(n-m-1)\phi\tan^{-1}\left(\frac{x\sin\phi}{1 - x\cos\phi}\right) - \frac{\sin(n-m-1)\phi}{2n\sin\theta}\log(x^2 - 2x\cos\phi + 1),$$

where $\phi = \dfrac{2r\pi + \theta}{n}$, and, to obtain all the terms, r must be taken from 0 to $n-1$.

CHAPTER III.

INTEGRATION BY RATIONALIZATION.

36. In the preceding Chapter we have shown how to obtain the value of the integral $\int u\,dx$, where u is a rational function of x; and we now come to the case when u involves irrational expressions. In several cases the irrational expressions can be rendered rational by certain transformations, so that the integration may be then considered as performed. In general this simplification cannot be effected except when u involves a radical not containing higher power of x than the second, and in certain other particular cases, which we proceed to consider.

37. If an algebraic expression involve fractional powers of the variable x, it can evidently be made rational by putting $x = y^m$, where m is the least common multiple of the denominators of the fractional powers. Thus, for instance, to integrate $\dfrac{dx}{x^{\frac{1}{2}}(1 + x^{\frac{1}{3}})}$, we put $x = y^6$, when the integral becomes

$$\int \frac{6y^2\,dy}{1 + y^2} = 6\,(y - \tan^{-1} y).$$

Hence we get

$$\int \frac{dx}{x^{\frac{1}{2}}(1 + x^{\frac{1}{3}})} = 6\,\{x^{\frac{1}{6}} - \tan^{-1}(x^{\frac{1}{6}})\}.$$

In the same way, if an algebraic expression involve integer powers of x, and fractional powers of $a + bx$, we put $a + bx = y^m$, and the reduction to a rational form is effected. More generally, if u involve integer powers of x and fractional powers of y, where

$$y = \frac{a + \beta x}{\gamma + \delta x},$$

it may be rationalized by putting $y = z^m$, where m has the same meaning as before; for we have

$$x = \frac{\gamma z^m - a}{\beta - \delta z^m},$$

and $$dx = \frac{(\beta\gamma - a\delta)\, m z^{m-1}\, dz}{(\beta - \delta z^m)^2}.$$

Hence we see that $u\,dx = v\,dz$, where v is a rational function of z. For example, to evaluate the integral

$$\int \sqrt{\left(\frac{x - a}{x - \beta}\right)} dx,$$

we put $$\frac{x - a}{x - \beta} = z^2,$$

whence $$x = \frac{a - \beta z^2}{1 - z^2},$$

and the integral becomes

$$2(a - \beta) \int \frac{z^2\, dz}{(1 - z^2)^2} = \frac{(a - \beta) z}{1 - z^2} - \frac{(a - \beta)}{2} \log\left(\frac{1 + z}{1 - z}\right)$$

$$= \sqrt{\{(x - a)(x - \beta)\}} - \tfrac{1}{2}(a - \beta) \log \left\{\frac{\sqrt{(x - \beta)} + \sqrt{(x - a)}}{\sqrt{(x - \beta)} - \sqrt{(x - \beta)}}\right\}.$$

EXAMPLES.

1. $$\int \frac{x^{\frac{1}{4}}\,dx}{1+x^{\frac{1}{2}}} = \tfrac{4}{3}x^{\frac{3}{4}} - 4x^{\frac{1}{4}} + 4\tan^{-1}(x^{\frac{1}{4}}).$$

2. $$\int \frac{dx}{x^{\frac{1}{3}}(1+x^{\frac{1}{4}})} = \frac{12}{5}x^{\frac{5}{12}} - 6x^{\frac{1}{6}} + 4\sqrt{3}\tan^{-1}\left(\frac{2x^{\frac{1}{12}}-1}{\sqrt{3}}\right)$$
$$- 2\log\left\{\frac{(1+x^{\frac{1}{12}})^2}{1-x^{\frac{1}{12}}+x^{\frac{1}{6}}}\right\}.$$

3. $$\int \frac{x^2\,dx}{\sqrt{(1+x)}} = \tfrac{2}{15}\sqrt{(1+x)}\,(3x^2-4x+8).$$

4. $$\int \frac{dx}{x^2\sqrt{(1+x)}} = -\frac{1}{x}\sqrt{(1+x)} + \frac{1}{2}\log\left\{\frac{\sqrt{(1+x)}+1}{\sqrt{(1+x)}-1}\right\}.$$

5. $$\int \frac{dx}{x^2-2x+4-4\sqrt{(1+x)}} = -\frac{1}{3}\,\frac{1}{\sqrt{(1+x)}-1}$$
$$+ \frac{1}{18}\log\left\{\frac{x+2-2\sqrt{(1+x)}}{x+4+2\sqrt{(1+x)}}\right\}$$
$$- \frac{5}{9\sqrt{2}}\tan^{-1}\left\{\frac{1+\sqrt{(1+x)}}{\sqrt{2}}\right\}.$$

6. $$\int \sqrt{\left(\frac{a-x}{x-b}\right)}\,dx = \sqrt{\{(a-x)(x-b)\}} - (a-b)\tan^{-1}\left\{\sqrt{\left(\frac{a-x}{x-b}\right)}\right\}.$$

7. $$\int \left(\frac{x-a}{x-b}\right)^{\frac{3}{2}} dx = \sqrt{\{(x-a)(x-b)\}}$$
$$+ 2(a-b)\sqrt{\left(\frac{x-a}{x-b}\right)} + \frac{3}{2}(a-b)\log\left\{\frac{\sqrt{(x-b)}-\sqrt{(x-a)}}{\sqrt{(x-b)}+\sqrt{(x-a)}}\right\}.$$

8. $$\int \left(\frac{x-a}{x-b}\right)^{\frac{1}{3}} dx = (x-a)^{\frac{1}{3}}(x-b)^{\frac{2}{3}} + \tfrac{1}{2}(a-b)\log\{(x-b)^{\frac{1}{3}}-(x-a)^{\frac{1}{3}}\}$$
$$- \frac{(a-b)}{\sqrt{3}}\tan^{-1}\left\{\frac{2(x-a)^{\frac{1}{3}}+(x-b)^{\frac{1}{3}}}{(x-b)^{\frac{1}{3}}\sqrt{3}}\right\}.$$

9. $\displaystyle\int \frac{dx}{x}\left(\frac{x-a}{x-b}\right)^{\frac{1}{3}} = 3\int \frac{dz}{1-z^3} - 3a\int \frac{dz}{a-bz^3},$

where $$x = \frac{a-bz^3}{1-z^3}.$$

10. $\displaystyle\int \frac{dx}{x}\sqrt{\left(\frac{a^2-x^2}{x^2-b^2}\right)} = \tan^{-1}\sqrt{\left(\frac{a^2-x^2}{x^2-b^2}\right)} - \frac{b}{a}\tan^{-1}\left\{\frac{a}{b}\sqrt{\left(\frac{a^2-x^2}{x^2-b^2}\right)}\right\}.$

38. We now come to the consideration of the integral

$$\int f(x, y)\, dx, \tag{1}$$

where $$y^2 = ax^2 + 2bx + c = X.$$

The most general form of $f(x, y)$ can be evidently written $(A + By)/(C + Dy)$, where A, B, C, D are rational integral expressions in x. Now this is equal to

$$\frac{(A + By)(C - Dy)}{C^2 - D^2y^2},$$

or $$\frac{AC - DBX}{C^2 - D^2X} + \frac{(BC - AD)X}{(C^2 - D^2X)y} = P + \frac{Q}{y}, \text{ say,}$$

where P, Q are rational fractions.

But $\int P dx$ comes under the case of rational functions, so that we have to consider

$$\int \frac{Q\,dx}{\sqrt{(ax^2 + 2bx + c)}}. \tag{2}$$

Our object is then to reduce the integration of this expression to that of a rational function.

Now if we put $y + h$ for x, we can evidently determine h so that $ax^2 + 2bx + c$ shall assume one or other of the three forms

$$a^2 - y^2, \quad y^2 - a^2, \quad y^2 + a^2.$$

But, from Art. 6, we know that the transformation

$$y = \frac{2az}{1+z^2}$$

will rationalize $\sqrt{(a^2-y^2)}$. In fact, we have

$$\sqrt{(a^2-y^2)} = a\frac{(1-z^2)}{1+z^2},$$

and
$$dy = 2a\frac{(1-z^2)\,dz}{(1+z^2)^2}.$$

Again, from Art. 14, we see that the transformation

$$y = \tfrac{1}{2}\left(z \pm \frac{a^2}{z}\right) \text{ rationalizes } \sqrt{(y^2 \mp a^2)},$$

in which case we have

$$\frac{dy}{\sqrt{(y^2 \mp a^2)}} = \frac{dz}{z}.$$

We see thus that in the three cases we are able to make (2) depend upon the integration of $u\,dz$, where u is a rational expression in z.

We might treat the integral (2) directly in the following manner also:—If a is positive, assume

$$ax^2 + 2bx + c = a(x+z)^2, \tag{3}$$

then
$$2bx + c = az^2 + 2azx,$$

or
$$2x = \frac{az^2 - c}{2(b-az)}. \tag{4}$$

Also $(b - az)\,dx = a(z+x)\,dz = \sqrt{\{a(ax^2+2bx+c)\}}\,dz;$

therefore
$$\frac{dx}{\sqrt{(ax^2+2bx+c)}} = \frac{\sqrt{a}\,dz}{b-az}. \tag{5}$$

We see thus that this substitution renders (2) rational. For example, the integral

$$\int \frac{dx}{(x-a)\sqrt{(ax^2+2bx+c)}}$$

becomes $$\int \frac{2\sqrt{a}\,dz}{az^2+2aaz-2ba-c} = \frac{1}{\Delta}\log\left\{\frac{\sqrt{a}\,(z+a)-\Delta}{\sqrt{a}\,(z+a)+\Delta}\right\},$$

where $$\Delta^2 = aa^2+2ba+c.$$

Again, if c is positive, assume

$$ax^2+2bx+c = c\,(1+xz)^2; \tag{6}$$

then $$ax+2b = c\,(xz^2+2z),$$

and $$x = \frac{2\,(cz-b)}{a-cz^2}. \tag{7}$$

Also $$\frac{dx}{\sqrt{(ax^2+2bx+c)}} = \frac{2\sqrt{c}\,dz}{a-cz^2}. \tag{8}$$

Thus we see that this substitution rationalizes (2) provided c be positive. For example, the integral

$$\int \frac{dx}{(x-a)\sqrt{(1-x^2)}},$$

becomes $$\int \frac{2\,dz}{a\,(1+z^2)+2z} = 2\int \frac{d\,(az+1)}{(az+1)^2+a^2-1},$$

where $$x = -\frac{2z}{1+z^2}.$$

We now mention a transformation corresponding to the case in which the roots of $ax^2+2bx+c$ are real. If a, β are the roots, it is easy to see that $ax^2+2bx+c$ will be proportional to one or other of the forms

$$(x-a)(x-\beta), \quad (a-x)(x-\beta).$$

Now $\sqrt{\{(x-a)(x-\beta)\}} = (x-\beta)\sqrt{\left(\frac{x-a}{x-\beta}\right)}$.

Hence, by the preceding Article, we put

$$\frac{x-a}{x-\beta} = z^2,$$

whence

$$x = \frac{a-\beta z^2}{1-z^2}, \tag{9}$$

and

$$\frac{dx}{\sqrt{\{(x-a)(x-\beta)\}}} = \frac{2dz}{1-z^2}. \tag{10}$$

In the second case we get, similarly,

$$x = \frac{a+\beta z^2}{1+z^2}, \tag{11}$$

whence we find

$$\frac{dx}{\sqrt{\{(a-x)(x-\beta)\}}} = \frac{2dz}{1+z^2}. \tag{12}$$

The preceding transformations include all the cases that can occur; for if a and c are both negative, it is easy to see that the roots of $ax^2 + 2bx + c$ must be real, so that then we make use of (11).

The integration of $u\,dx$, where u is a rational function of x and the square roots of two linear expressions in x, can be reduced to the preceding cases. Let

$$u = f\left(x, \sqrt{(a+bx)}, \sqrt{(a'+b'x)}\right);$$

then, putting $a + bx = y^2$, we have

$$u = \phi\left(y, \sqrt{(a'b - b'a + b'y^2)}\right),$$

and $b\,dx = 2y\,dy$, which proves the statement just made.

EXAMPLES.

1. $$\int \frac{(l + mx)\,dx}{\{(x-a)^2 + \beta^2\}\sqrt{(ax^2 + 2bx + c)}}$$
$$= \int \frac{2\sqrt{a}\,\{m(az^2 - c) + 2l(b - az)\}\,dz}{\{az^2 - c - 2\alpha(b - az)\}^2 + 4\beta^2(b - az)^2},$$

where $$x = \frac{az^2 - c}{2(b - az)};$$

or $$\int \frac{2\sqrt{c}\,\{l(a - cz^2) + 2m(cz - b)\}\,dz}{\{\alpha(a - cz^2) - 2(cz - b)\}^2 + \beta^2(a - cz^2)^2},$$

where $$x = \frac{2(cz - b)}{a - cz^2}.$$

2. $$\int \frac{dx}{(x - a)\sqrt{(1 - x^2)}} = \int \frac{2dz}{(1 - a)z^2 - (1 + a)},$$

where $$x = \frac{z^2 - 1}{z^2 + 1}.$$

3. $$\int \frac{x^n\,dx}{\sqrt{(ax^2 + 2bx + c)}} = \frac{\sqrt{a}}{2^n}\int \frac{(az^2 - c)^n\,dz}{(b - az)^{n+1}},$$

where $$x = \frac{az^2 - c}{2(b - az)};$$

putting, then, $b - az = y$, we get

$$\int \frac{(az^2 - c)^n\,dz}{(b - az)^{n+1}} = -\frac{1}{a^{n+1}}\int \frac{dy}{y^{n+1}}\,(y^2 - 2by + b^2 - ac)^n,$$

which may be integrated at once by expanding $(y^2 - 2by + b^2 - ac)^n$.

4. $$\int \frac{dx}{x^n\sqrt{(1 - x^2)}} = \frac{1}{2^{n-1}}\int \frac{dz}{z^n}\,(1 + z^2)^{n-1},$$

where $$x = \frac{2z}{1 + z^2}.$$

5. $$\int \{x + \sqrt{(x^2 + a^2)}\}^n \frac{dx}{\sqrt{(x^2 + a^2)}} = \frac{1}{n}\{x + \sqrt{(x^2 + a^2)}\}^n.$$

6. $$\int \{x + \sqrt{(x^2 + a^2)}\}^n\,dx = \frac{\{x + \sqrt{(x^2 + a^2)}\}^n}{n^2 - 1}\{n\sqrt{(x^2 - a^2)} - x\}.$$

7. $\displaystyle\int \frac{dx}{x}\{x+\sqrt{(x^2+a^2)}\}^{\frac{1}{2}} = 2\sqrt{z}$

$$-\sqrt{\left(\frac{a}{2}\right)}\log\left\{\frac{a+z+\sqrt{(2az)}}{a+z-\sqrt{(2az)}}\right\}-\sqrt{(2a)}\tan^{-1}\left\{\frac{\sqrt{(2az)}}{a-z}\right\},$$

where $$z = x+\sqrt{(x^2+a^2)}.$$

8. $\displaystyle\int \frac{dx}{\{m\sqrt{(\alpha-x)}+n\sqrt{(x-\beta)}\}\sqrt{\{(\alpha-x)(x-\beta)\}}}$

$$=2\sqrt{\left(\frac{\alpha-\beta}{m^2+n^2}\right)}\log\left\{\frac{\sqrt{\{(\alpha-\beta)(m^2+n^2)\}}+m\sqrt{(x-\beta)}-n\sqrt{(\alpha-x)}}{m\sqrt{(\alpha-x)}+n\sqrt{(x-\beta)}}\right\}.$$

9. $\displaystyle\int \{\sqrt{x}+\sqrt{(x-a)}\}^{\frac{1}{2}}\frac{(\sqrt{x}-\sqrt{a})\,dx}{(x-a)^{\frac{3}{2}}\sqrt{x}}$

$$=\frac{2}{(a)^{\frac{1}{2}}}\left\{\sin^{-1}\left\{\frac{\sqrt{x}-\sqrt{a}}{\sqrt{(x-a)}}\right\}-\left\{\frac{2\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right\}^{\frac{1}{2}}\right\}.$$

39. We now proceed to consider the general transformation which serves to rationalize the expression involved in the integral (2). For the sake of symmetry we shall render (2) homogeneous, by putting x/y for x, when it may be written in the form

$$\int \frac{\phi(x, y)\,(x\,dy - y\,dx)}{\sqrt{(ax^2+2bxy+cy^2)}}, \tag{13}$$

where $\phi(x, y)$ is a homogeneous expression in x, y of the degree -1. Let us put now P, Q for x, y, respectively, where

$$P = \alpha x^2 + 2\beta xy + \gamma y^2,$$
$$Q = \alpha' x^2 + 2\beta' xy + \gamma' y^2;$$

then the integral just written becomes

$$\tfrac{1}{2}\int \frac{\phi(P, Q)\,J\,(x\,dy - y\,dx)}{\sqrt{(aP^2+2bPQ+cQ^2)}}, \tag{14}$$

where J has the same meaning as in Art. 21.

Now this expression will evidently be rational if we determine the quantities α, β, γ, α', β', γ', so that

$$aP^2 + 2bPQ + cQ^2$$

should be a perfect square. To effect this, we shall obtain a quadratic expression in P, Q, which is a perfect square, when it is expressed as a function of x, y. Now, if we take the condition that $P - \lambda Q$ should be a perfect square in x, y, we have

$$\lambda^2(\alpha'\gamma' - \beta'^2) - \lambda(\alpha'\gamma + \gamma'\alpha - 2\beta\beta') + \alpha\gamma - \beta^2 = 0; \quad (15)$$

so that if λ_1, λ_2 are the roots of this equation in λ, we shall evidently have $(P - \lambda_1 Q)(P - \lambda_2 Q)$ equal to the square of a quadratic expression in x, y. Hence, in order that

$$aP^2 + 2bPQ + cQ^2$$

should be a perfect square, we must express that it coincides with the result of eliminating λ between (15) and $P - \lambda Q = 0$. We thus find

$$\frac{\alpha'\gamma' - \beta'^2}{a} = -\frac{(\alpha'\gamma + \gamma'\alpha - 2\beta\beta')}{2b} = \frac{\alpha\gamma - \beta^2}{c}; \quad (16)$$

and these are the only conditions to which α, β, γ, α', β', γ' must be subjected in order that (14) should become rational.

To find the quadratic whose square is proportional to $aP^2 + 2bPQ + cQ^2$, we observe that one of its factors is a square factor of $P - \lambda_1 Q$, and therefore must satisfy the result of equating to zero the differential coefficients of this expression, namely,

$$\frac{dP}{dx} - \lambda_1\frac{dQ}{dx} = 0, \quad \frac{dP}{dy} - \lambda_1\frac{dQ}{dy} = 0;$$

whence, eliminating λ, we get

$$\frac{dP}{dx}\frac{dQ}{dy} - \frac{dP}{dy}\frac{dQ}{dx} = J = 0.$$

Hence we infer that, subject to the conditions (16), $aP^2 + 2bPQ + cQ^2$ is proportional to J^2. We see thus that J disappears altogether from (14), which becomes proportional to

$$\int \phi(P, Q)(x\,dy - y\,dx). \qquad (17)$$

40. We can now prove the statement made in Art. 20, namely, that the integral

$$\int \frac{x\,dy - y\,dx}{v\sqrt{u}}$$

can be transformed by a quadratic substitution to the form

$$\int \frac{x\,dy - y\,dx}{w},$$

where u, w are quadratics, and v is linear in x, y.

Let $\phi(x, y)$ in (13) be taken equal to v^{-1}, where $v = lx + my$; then, by the substitution made use of above, the given integral is, from (17), transformed into

$$\int \frac{x\,dy - y\,dx}{lP + mQ},$$

which is of the form of the second of the two integrals just mentioned.

41. From the preceding general transformation we can derive several particular cases by assuming certain relations between the constants α, β, γ, α', β', γ'. For instance, if we take $\beta = \alpha' = 0$, $\alpha = a$, $\beta' = -a$; then from (15) we find

$\gamma = -c$, $\gamma' = 2b$, so that for x/y we are to put

$$\frac{ax^2 - cy^2}{2(by^2 - axy)};$$

but this is evidently the transformation (4) already made use of.

Again, taking $\alpha = \beta' = 0$, $\alpha' = -c$, $\beta = c$,

we get $$\gamma = -2b, \quad \gamma' = a,$$

which gives the transformation (7).

If we take $\beta = \beta' = 0$, $\alpha' = \gamma' = 1$,

we find from (15), that α, γ are the roots of the equation

$$a\alpha^2 + 2b\alpha + c = 0,$$

so that for x/y we put then

$$\frac{\alpha x^2 + \gamma y^2}{x^2 + y^2};$$

or, changing the sign of y^2,

$$\frac{\alpha x^2 - \gamma y^2}{x^2 - y^2}.$$

These are evidently the transformations (9) and (11).

42. We have already remarked that if u involves the square root of an expression which contains powers of x of the third or fourth or higher degrees, the integral of $u\,dx$ will depend upon corresponding transcendental functions. In certain particular cases, however, we can reduce to elementary forms the integral

$$\int \frac{R\,dx}{\sqrt{X}},$$

where R is a rational function of x, and X contains powers

of the same variable as far as the fourth degree. Putting x/y for x, this integral may be written

$$\int \frac{f(x, y)(x\,dy - y\,dx)}{\sqrt{X}}, \tag{18}$$

where $\qquad X = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$

and $f(x, y)$ is the ratio of two homogeneous integral expressions in x, y of the same degree.

Now, from (14), we see at once that

$$\int \frac{\phi(P, Q)\,J(x\,dy - y\,dx)}{\sqrt{(PQ)}}$$

depends upon circular or logarithmic functions. It follows, hence, that (18) will be reducible to the elementary integrals, if $f(x, y)$ is of the form $J\phi(P, Q)$, where P, Q are a pair of quadratic factors of X; J is the Jacobian of P, Q, and $\phi(P, Q)$ is a rational homogeneous expression in P, Q of the degree -1.

For example, let $\quad P = x^2 + y^2 + 2axy,$

$$Q = x^2 + y^2 + 2bxy,$$

then J is proportional to $x^2 - y^2$, and the integral

$$\int \frac{\phi(x^2 + y^2, xy)(x^2 - y^2)(x\,dy - y\,dx)}{\sqrt{\{(x^2 + y^2 + 2axy)(x^2 + y^2 + 2bxy)\}}}$$

is reducible to the elementary forms.

Again, we can show in the same way that the integral

$$\int \frac{\phi(x^2 - y^2, xy)(x^2 + y^2)(x\,dy - y\,dx)}{\sqrt{\{(x^2 - y^2 + 2axy)(x^2 - y^2 + 2bxy)\}}}$$

admits of a similar reduction.

43. In several other exceptional cases expressions involving a radical can be rationalized. For instance, the integral

$$\int \frac{dx}{x}(a + bx^n)^{\frac{p}{q}}$$

becomes

$$\frac{q}{n}\int \frac{z^{p+q-1}\,dz}{z^q - a},$$

if we put

$$a + bx^n = z^q.$$

Again,

$$\int \frac{x^{m-1}\,dx}{(a + bx^n)^{\frac{m}{n}}}$$

is transformed into

$$\int \frac{z^{n-m-1}\,dz}{b - z^n},$$

where

$$a + bx^n = (xz)^n.$$

Examples.

1. $$\int \frac{x\,dx}{(a + x)^{\frac{1}{2}} + (a + x)^{\frac{1}{4}}} = 4\int \frac{dy}{y}\{(y - 1)^6 - a^2(y - 1)^2\}$$

where

$$y = 1 + (a + x)^{\frac{1}{4}}.$$

2. $$\int \frac{x^2\,dx}{(1 + x)^{\frac{1}{3}}} = \tfrac{3}{40}(5x^2 - 6x + 9)(1 + x)^{\frac{2}{3}}.$$

3. $$\int \frac{x^2\,dx}{(1 + x)^{\frac{2}{3}}} = \tfrac{3}{14}(2x^2 - 3x + 9)(1 + x)^{\frac{1}{3}}.$$

4. $$\int \left(\frac{x - a}{x - b}\right)^{\frac{1}{3}} dx = (x - a)^{\frac{1}{3}}(x - b)^{\frac{2}{3}} + (a - b)\{(x - b)^{\frac{1}{3}} - (x - a)^{\frac{1}{3}}\}$$

$$+ \frac{2}{\sqrt{3}}(a - b)\tan^{-1}\left\{\frac{2(x - a)^{\frac{1}{3}} + (x - b)^{\frac{1}{3}}}{(x - b)^{\frac{1}{3}}\sqrt{3}}\right\}.$$

5. $$\int \frac{(x-a)^{\frac{1}{2}}\,dx}{(x-b)^{\frac{3}{2}}} = \log\left\{\frac{\sqrt{(x-b)}+\sqrt{(x-a)}}{\sqrt{(x-b)}-\sqrt{(x-a)}}\right\} - 2\left(\frac{x-a}{x-b}\right)^{\frac{1}{2}}.$$

6. $$\int \frac{dx}{(x-a)^{\frac{3}{4}}(x-b)^{\frac{1}{4}}} = \log\left\{\frac{(x-b)^{\frac{1}{4}}+(x-a)^{\frac{1}{4}}}{(x-b)^{\frac{1}{4}}-(x-a)^{\frac{1}{4}}}\right\} + 2\tan^{-1}\left\{\frac{x-a}{x-b}\right\}^{\frac{1}{4}}.$$

7. $$\int \frac{dx}{x^n\sqrt{(ax^2+2bx+c)}} = \frac{\sqrt{c}}{2^{n-1}}\int\frac{(a-cz^2)^{n-1}}{(cz-b)^n}\,dz$$

$$= \frac{1}{2^{n-1}\,c^{n-\frac{1}{2}}}\int\frac{dy}{y^n}\,(ac-b^2-2by-y^2)^{n-1},$$

where $$x = \frac{2(cz-b)}{a-cz^2}, \quad cz = y+b.$$

8. $$\int x^m\{x+\sqrt{(1+x^2)}\}^n dx = \frac{1}{2^{m+1}}\int\frac{(z^2-1)^m}{z^{m-n}}\,dz + \frac{1}{2^{m+1}}\int\frac{(z^2-1)^m}{z^{m-n+2}}\,dz,$$

where $$x+\sqrt{(1+x^2)} = z.$$

9. $$\int x^m\{x+\sqrt{(1+x^2)}\}^n\frac{dx}{\sqrt{(1+x^2)}} = \frac{1}{2^m}\int\frac{(z^2-1)^m}{z^{m-n+1}}\,dz,$$

where $$x+\sqrt{(1+x^2)} = z.$$

10. $$\int \frac{dx}{\sqrt{\{(\alpha-x)(x-\beta)\}}}\;\frac{1}{\{\sqrt{(\alpha-x)}+m\sqrt{(x-\beta)}\}}$$

$$= \frac{-4}{\sqrt{(\alpha-\beta)}}\int\frac{dy}{1-y^2+2my},$$

where $$x = \frac{\alpha(1-y^2)^2+4\beta y^2}{(1+y^2)^2}.$$

11. $$\int \frac{\{\sqrt{(\alpha-x)}+m\sqrt{(x-\beta)}\}^n\,dx}{\sqrt{\{(\alpha-x)(x-\beta)\}}} = -4\int\frac{(1-y^2+2my)^n}{(1+y^2)^{n+1}}\,dy.$$

12. $$\int \{\sqrt{(\alpha-x)}+m\sqrt{(x-\beta)}\}^n dx = -8(\alpha-\beta)\int\frac{(1-y^2+2my)^n}{(1+y^2)^{n+3}}$$

$$\times(1-y^2)\,y\,dy.$$

13. $$\int \frac{(x^2-1)\,dx}{(x^2+1+2\gamma x)\sqrt{\{(x^2+1+2\alpha x)(x^2+1+2\beta x)\}}}$$
$$= \int \frac{dz}{\gamma-\beta+(\alpha-\gamma)z^2},$$
where $$\frac{x^2+1+2\beta x}{x^2+1+2\alpha x} = z^2.$$

14. $$\int \frac{(x^2+1)\,dx}{(x^2-1+2\gamma x)\sqrt{\{(x^2-1+2\alpha x)(x^2-1+2\beta x)\}}}$$
$$= \int \frac{dz}{\gamma-\beta+(\alpha-\gamma)z^2},$$
where $$\frac{x^2-1+2\beta x}{x^2-1+2\alpha x} = z^2.$$

15. $$\int \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{1+x^2}\right).$$

16. $$\int \frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\log\left(\frac{x^2-1}{x\sqrt{2}+\sqrt{(1+x^4)}}\right).$$

17. $$\int \frac{dx}{1-x^4}\sqrt{(x^4+2x^2\cos 2\alpha+1)}$$
$$= \frac{1}{2}\sin\alpha\sin^{-1}\left(\frac{2x\sin\alpha}{1+x^2}\right) + \frac{1}{2}\cos\alpha\log\frac{2x\cos\alpha+\sqrt{(x^4+2x^2\cos 2\alpha+1)}}{1-x^2}.$$

18. $$\int \frac{1-x^2}{1+x^2}\frac{dx}{\sqrt{(x^4+2x^2\cos 2\alpha+1)}} = \frac{1}{2\sin\alpha}\sin^{-1}\left(\frac{2x\sin\alpha}{1+x^2}\right).$$

19. $$\int \frac{x\,dx}{(1+x^3)^{\frac{2}{3}}} = \int \frac{dz}{1-z^3}, \quad \text{where } 1+x^3 = x^3z^3.$$

20. $$\int \frac{dx}{(x^4-1)^{\frac{1}{4}}} = \frac{1}{4}\log\frac{1+z}{1-z} - \frac{1}{2}\tan^{-1}z, \quad \text{where } z = \frac{1}{x}(x^4-1)^{\frac{1}{4}}.$$

21. $$\int \frac{dx}{(1+x^n)(1+2x^n)^{\frac{1}{2n}}} = \int \frac{dz}{1+z^{2n}}, \quad \text{where } z^{2n}(1+2x^n) = x^{2n}.$$

CHAPTER IV.

INTEGRATION BY SUCCESSIVE REDUCTION.

44. THE method employed in this Chapter is one of the most important in the Integral Calculus, and is not only applicable to algebraic differentials, but also to many expressions involving transcendents. Its principal use is, however, to reduce the integral $\int f(x, X^p)\,dx$ to certain standard forms, where X is a rational integral expression in x, and p is some number, whole or fractional, positive or negative. Under this form are included such integrals as

$$\int x^m X^p\,dx, \tag{1}$$

where m is a positive or negative integer such as p. The object, then, of this method is by means of a formula of reduction to make the integral (1) depend upon another or others of the same form, in which one, or other, or both, of the numbers m, p are diminished or increased; so that by a repetition of the formula we can ultimately express (1) in terms of the simplest integrals of the series.

The easiest procedure in practice is frequently to assume some function of x, such that its differential can be expressed in terms of those of two or more integrals of the same form. By integration, then, we have these integrals connected

with an algebraic quantity, and this constitutes a formula of reduction. For example, differentiating

$$x^m X^p, \text{ where } X = a + bx^n,$$

we have

$$\frac{d}{dx}(x^m X^p) = x^{m-1} X^{p-1}(mX + npbx^n)$$
$$= (m + np)x^{m-1}X^p - npax^{m-1}X^{p-1}; \quad (2)$$

whence, by integration, we find the integrals

$$\int x^{m-1} X^p \, dx \quad \text{and} \quad \int x^{m-1} X^{p-1} \, dx$$

connected. Again, we have

$$\frac{d}{dx}(x^m X^p) = max^{m-1}X^{p-1} + (m + np)bx^{m+n-1}X^{p-1},$$

from which we find the integrals

$$\int x^{m-1} X^{p-1} \, dx \quad \text{and} \quad \int x^{m+n-1} X^{p-1} \, dx$$

connected. We see thus at once that we can connect the integral $\int x^{m-1} X^p \, dx$ with any one of the four following integrals:

$$\int x^{m-1} X^{p-1} dx, \quad \int x^{m-1} X^{p+1} dx, \quad \int x^{m-n-1} X^p dx, \quad \int x^{m+n-1} X^p dx;$$

and we can select then whichever of these is the most appropriate, according to the values of m, p, n. When this is done, we take the one of lower dimensions of the two expressions whose integrals are to be connected, increase the indices of both x and X by unity, and assume the result equal to P. Then dP can be resolved, as has been shown above, into the sum of the two expressions whose integrals are to be connected, whence by integration the formula of reduction is

obtained, which, by successive applications, serves to reduce the given integral to a known form, or the simplest integral of the series.

45. In many cases the integrals of rational expressions of the variable are most easily treated by the method of this chapter. We commence with the integral

$$\int \frac{dx}{X^n},$$

where $X = a + 2bx + cx^2$, and n is a positive integer. Taking

$$P = \frac{b + cx}{X^{n-1}},$$

we have

$$\frac{dP}{dx} = \frac{c}{X^{n-1}} - 2(n-1)\frac{(b+cx)^2}{X^n}$$

$$= 2(n-1)\frac{(ac-b^2)}{X^n} - \frac{(2n-3)c}{X^{n-1}}.$$

Hence, integrating, we get

$$\int \frac{dx}{X^n} = \frac{b+cx}{2(n-1)(ac-b^2)X^{n-1}} + \frac{(2n-3)c}{2(n-1)(ac-b^2)}\int \frac{dx}{X^{n-1}}. \quad (3)$$

By the aid of this formula of reduction, the integral $\int dx / X^n$ is made to depend upon $\int dx / X$, which is found from (12) or (13) of Chap. I.

By the substitution of $y + h$ for x, we can remove the coefficient of y, and we may then more readily separate the cases in which the factors of X are real and imaginary. If $X = a^2 - x^2$, we have

$$\int \frac{dx}{X^n} = \frac{x}{(2n-2)a^2X^{n-1}} + \frac{2n-3}{(2n-2)a^2}\int \frac{dx}{X^{n-1}}. \quad (4)$$

Hence, changing n successively into $n-1$, $n-2$, &c., we get

$$\int \frac{dx}{X^{n-1}} = \frac{x}{(2n-4)a^2 X^{n-2}} + \frac{2n-5}{(2n-4)a^2} \int \frac{dx}{X^{n-2}} \text{ \&c.,}$$

till at last we come to

$$\int \frac{dx}{X} = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right);$$

so that we have

$$\int \frac{dx}{X^n} = \frac{x}{X^{n-1}} \left\{ \frac{1}{(2n-2)a^2} + \frac{(2n-3)}{(2n-2)(2n-4)} \frac{X}{a^4} \right.$$

$$+ \frac{(2n-3)(2n-5)}{(2n-2)(2n-4)(2n-6)} \frac{X^2}{a^6} + \ldots$$

$$\left. + \frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2} \frac{X^{n-2}}{a^{2n-2}} \right\}$$

$$+ \frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2} \frac{1}{2a^{2n-1}} \log\left(\frac{a+x}{a-x}\right). \qquad (5)$$

Again, if $X = a^2 + x^2$, we get a formula of reduction of exactly the same form as (4) for the integral $\int dx / X^n$. We find then an expression of the same form as (5) for this integral, with the exception that the last term is replaced by

$$\frac{(2n-3)(2n-5)\ldots 3.1}{(2n-2)(2n-4)\ldots 4.2} \frac{1}{a^{2n-1}} \tan^{-1}\frac{x}{a}. \qquad (6)$$

46. We now consider the more general integral

$$\int \frac{x^m\, dx}{(a + 2bx + cx^2)^n}. \qquad (7)$$

If $a + 2bx + cx^2 = X$, we have

$$\frac{d}{dx}\left(\frac{x^{m-1}}{X^{n-1}}\right) = \frac{(m-1)x^{m-2}}{X^{n-1}} - \frac{2(n-1)x^{m-1}(b+cx)}{X^n}$$

$$= \frac{(m-1)ax^{m-2}}{X^n} + \frac{2b(m-n)x^{m-1}}{X^n} - \frac{(2n-m-1)cx^m}{X^n};$$

Hence, by integration, we obtain

$$\int \frac{x^m\,dx}{X^n} = \frac{-x^{m-1}}{c(2n-m-1)X^{n-1}} + \frac{2b(m-n)}{c(2n-m-1)}\int \frac{x^{m-1}\,dx}{X^n}$$

$$+ \frac{a(m-1)}{c(2n-m-1)}\int \frac{x^{m-2}\,dx}{X^n}, \qquad (8)$$

which gives a formula of reduction, by successive applications of which we can make (7) depend upon the integrals

$$\int x\,dx/X^n, \quad \int dx/X^n.$$

Now we have

$$\int \frac{x\,dx}{X^n} = \frac{1}{c}\int \frac{(b+cx)\,dx}{X^n} - \frac{b}{c}\int \frac{dx}{X^n}$$

$$= \frac{-1}{2c(n-1)X^{n-1}} - \frac{b}{c}\int \frac{dx}{X^n}.$$

We see thus that (7) is ultimately made to depend upon the integral considered in the preceding Article.

47. The integral $\displaystyle\int \frac{x^m\,dx}{(a^2 \pm x^2)^n}$

evidently comes under the preceding case, but may be more

easily evaluated by means of a different formula of reduction. Considering the integral

$$\int \frac{x^m\,dx}{(a+cx^2)^n}, \tag{9}$$

which includes both signs, we have

$$\int \frac{x^m\,dx}{(a+cx^2)^n} = \int \frac{x^{m-1}.\,x\,dx}{(a+cx^2)^n}$$

$$= \frac{-1}{2c(n-1)} \int x^{m-1}\,d\left\{\frac{1}{(a+cx^2)^{n-1}}\right\}.$$

Hence, integrating by parts, we find

$$\int \frac{x^m\,dx}{(a+cx^2)^n} = \frac{-x^{m-1}}{2c(n-1)(a+cx^2)^{n-1}}$$

$$+ \frac{m-1}{2c(n-1)} \int \frac{x^{m-2}\,dx}{(a+cx^2)^{n-1}}. \tag{10}$$

The successive applications of this formula of reduction evidently simplify the given integral more rapidly than those of (8).

EXAMPLES.

1. $$\int \frac{dx}{(x^2-2x\cos\theta+1)^2} = \frac{x-\cos\theta}{2\sin^2\theta(x^2-2x\cos\theta+1)}$$

$$+ \frac{1}{2\sin^3\theta}\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right).$$

2. $$\int \frac{dx}{(x^2-2x\cos\theta+1)^3} = \frac{x-\cos\theta}{4\sin^2\theta(x^2-2x\cos\theta+1)^2}$$

$$+ \frac{3}{8\sin^4\theta}\frac{(x-\cos\theta)}{(x^2-2x\cos\theta+1)} + \frac{3}{8\sin^5\theta}\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right).$$

3. $$\int \frac{dx}{(a^2 - x^2)^3} = \frac{x(5a^2 - 3x^2)}{8a^4(a^2 - x^2)^2} + \frac{3}{16a^5} \log\left(\frac{a + x}{a - x}\right).$$

4. $$\int \frac{dx}{(a^2 + x^2)^3} = \frac{x(5a^2 + 3x^2)}{8a^4(a^2 + x^2)^2} + \frac{3}{8a^5} \tan^{-1}\left(\frac{x}{a}\right).$$

5. $$\int \frac{x^2 dx}{(a + 2bx + cx^2)^2} = \frac{ab + (2b^2 - ac)x}{2c(ac - b^2)(a + 2bx + cx^2)} + \frac{a}{2(ac - b^2)} \int \frac{dx}{a + 2bx + cx^2}.$$

6. $$\int \frac{x^2 dx}{(1 + x^2)^4} = -\frac{1}{6}\frac{x}{(1 + x^2)^3} + \frac{1}{24}\frac{x}{(1 + x^2)^2} + \frac{1}{16}\frac{x}{1 + x^2} + \frac{1}{16}\tan^{-1}x.$$

7. $$\int \frac{x^4 dx}{(1 + x^2)^5} = -\frac{1}{8}\frac{x^3}{(1 + x^2)^4} - \frac{1}{16}\frac{x}{(1 + x^2)^3} + \frac{1}{64}\frac{x}{(1 + x^2)^2} + \frac{3}{128}\frac{x}{1 + x^2} + \frac{3}{128}\tan^{-1}x.$$

8. $$\int \frac{x^5 dx}{(1 - x^2)^3} = \frac{1}{4}\frac{1}{(1 - x^2)^2} - \frac{1}{1 - x^2} - \frac{1}{2}\log(1 - x^2).$$

9. $$\int \frac{x^5 dx}{(1 + x^2)^6} = \frac{1}{4}\frac{1}{(1 + x^2)^4} - \frac{1}{6}\frac{1}{(1 + x^2)^3} - \frac{1}{10}\frac{1}{(1 + x^2)^5}.$$

48. We now come to the consideration of the integral of the form $\int f(x, y)\,dx$, where $y^2 = a + 2bx + cx^2$. We have already seen in Art. 38 that such an integral depends upon another of the form

$$\int \frac{Q\,dx}{\sqrt{(a + 2bx + cx^2)}},$$

where Q is a rational fraction.

Now, if the numerator of Q is of a higher degree than the denominator, we may put

$$Q = R + \frac{S}{\phi(x)},$$

where R is a rational integral expression, and S is of a lower

degree than $\phi(x)$, the denominator of Q. Hence (11) becomes

$$\int \frac{R\,dx}{\sqrt{(a+2bx+cx^2)}} + \int \frac{S}{\phi(x)} \frac{dx}{\sqrt{(a+2bx+cx^2)}}. \qquad (12)$$

Now the general term in the first of these integrals is of the form

$$\int \frac{x^m\,dx}{\sqrt{(a+2bx+cx^2)}}, \qquad (13)$$

and for this expression we propose to investigate a formula of reduction. Taking

$$P = x^{m-1}\sqrt{(a+2bx+cx^2)},$$

we have

$$\frac{dP}{dx} = (m-1)\,x^{m-2}\sqrt{(a+2bx+cx^2)} + \frac{x^{m-1}(b+cx)}{\sqrt{(a+2bx+cx^2)}}$$

$$= \frac{(m-1)\,ax^{m-2} + (2m-1)\,bx^{m-1} + m\,cx^m}{\sqrt{(a+2bx+cx^2)}}.$$

Hence, by integration, we find

$$\int \frac{x^m\,dx}{\sqrt{(a+2bx+cx^2)}} = \frac{x^{m-1}}{mc}\sqrt{(a+2bx+cx^2)}$$

$$- \frac{(2m-1)\,b}{mc}\int \frac{x^{m-1}\,dx}{\sqrt{(a+2bx+cx^2)}} - \frac{(m-1)\,a}{mc}\int \frac{x^{m-2}\,dx}{\sqrt{(a+2bx+cx^2)}}. \qquad (14)$$

By means of this formula of reduction we ultimately make (13) depend upon the integrals

$$\int \frac{x\,dx}{\sqrt{(a+2bx+cx^2)}}, \quad \int \frac{dx}{\sqrt{(a+2bx+cx^2)}}.$$

Now we have

$$\int \frac{x\,dx}{\sqrt{(a+2bx+cx^2)}} = \frac{1}{c}\int \frac{(b+cx-b)\,dx}{\sqrt{(a+2bx+cx^2)}} = \frac{1}{c}\sqrt{(a+2bx+cx^2)}$$

$$-\frac{b}{c}\int \frac{dx}{\sqrt{(a+2bx+cx^2)}},$$

and the latter integral is given in Art. 15.

49. Proceeding now to the second part of (12), namely,

$$\int \frac{S}{\phi(x)} \frac{dx}{\sqrt{(a+2bx+cx^2)}}, \tag{15}$$

we can decompose $S/\phi(x)$ into partial fractions by the method of Chapter II. Hence, to every real factor $x - a$ of $\phi(x)$, which occurs once only, there is a term of the form

$$\int \frac{dx}{(x-a)\sqrt{(a+2bx+cx^2)}},$$

which has been already evaluated in Art. 16.

Again, to a multiple factor $(x-a)^p$ there correspond terms of the form

$$\int \frac{dx}{(x-a)^r\sqrt{(a+2bx+cx^2)}}.$$

But putting $x = a + z^{-1}$, this becomes

$$\int \frac{z^{r-1}\,dz}{\sqrt{(a'+2b'z+c'z^2)}},$$

where $\quad a' = c, \quad b' = b + ca, \quad c' = a + 2ba + ca^2,$

so that the integral thus assumes same form as (13), and is, therefore, reduced in the same way.

50. To a pair of conjugate imaginary roots of $\phi(x)$ will correspond an integral of the form

$$\int \frac{(lx+m)\,dx}{\{(x-a)^2+\beta^2\}\sqrt{(a+2bx+cx^2)}},$$

which has been already evaluated in Art. 16.

Again, if $\phi(x)$ has a multiple factor $\{(x-a)^2+\beta^2\}^q$, we see, from Art. 32, that there will be terms in the integral of the form

$$\int \frac{(Lx+M)\,dx}{\{(x-a)^2+\beta^2\}^r\sqrt{(a+2bx+cx^2)}};$$

or say, $$\int \frac{(lz+m)\,dz}{(z^2+\beta^2)^r\sqrt{(a'+2b'z+c'z^2)}}, \qquad (16)$$

if we put $x=a+z$.

We proceed now to investigate a formula of reduction for this integral. Dropping the accents for convenience, and putting $Z^2=a+2bz+cz^2$, we take

$$P=\frac{Z}{(z^2+\beta^2)^{r-1}};$$

and then we have

$$\frac{dP}{dz}=\frac{b+cz}{(z^2+\beta^2)^{r-1}Z}-\frac{2(r-1)zZ^2}{(z^2+\beta^2)^rZ}$$

$$=\frac{(b+cz)(z^2+\beta^2)-2(r-1)z(a+2bz+cz^2)}{(z^2+\beta^2)^rZ}$$

$$=\frac{2(r-1)\{(c\beta^2-a)z+2b\beta^2\}-\{(2r-3)cz+(4r-5)b\}(z^2+\beta^2)}{(z^2+\beta^2)^rZ}$$

Hence, by integration, we get

$$\int \frac{\{(c\beta^2 - a)z + 2b\beta^2\}\,dz}{(z^2+\beta^2)^r Z} = \frac{Z}{2(r-1)(z^2+\beta^2)^{r-1}}$$

$$+ \frac{1}{2(r-1)} \int \frac{\{(2r-3)cz + (4r-5)b\}\,dz}{(z^2+\beta^2)^{r-1} Z}. \qquad (17)$$

Again, taking $\qquad Q = \dfrac{zZ}{(z^2+\beta^2)^{r-1}},$

we have

$$\frac{dQ}{dz} = \frac{a + 3bz + 2cz^2}{(z^2+\beta^2)^{r-1} Z} - \frac{2(r-1)z^2 Z}{(z^2+\beta^2)^r}$$

$$= \frac{1}{(z^2+\beta^2)^r Z}\{(a+3bz+2cz^2)(z^2+\beta^2) - 2(r-1)z^2(a+2bz+cz^2)\}$$

$$= \frac{1}{(z^2+\beta^2)^r Z}\Big(2(r-1)\beta^2(a - c\beta^2 + 2bz) - (z^2+\beta^2)\{(4r-7)bz$$

$$+ (2r-3)(a - 2c\beta^2)\} - 2(r-2)c(z^2+\beta^2)^2\Big)$$

$$= \frac{2(r-1)\beta^2(a - c\beta^2 + 2bz)}{(z^2+\beta^2)^r Z} - \frac{(4r-7)bz + (2r-3)(a-2c\beta^2)}{(z^2+\beta^2)^{r-1} Z}$$

$$- \frac{2(r-2)c}{(z^2+\beta^2)^{r-2} Z}.$$

Integrating then, and restoring the value of Q, we get

$$\int \frac{(a - c\beta^2 + 2bz)\,dz}{(z^2+\beta^2)^r Z} = \frac{zZ}{2(r-1)\beta^2(z^2+\beta^2)^{r-1}}$$

$$+ \frac{1}{2(r-1)\beta^2}\int \frac{\{(4r-7)bz + (2r-3)(a-2c\beta^2)\}\,dz}{z^2+\beta^2)^{r-1} Z}$$

$$+ \frac{(r-2)c}{(r-1)\beta^2}\int \frac{dz}{(z^2+\beta^2)^{r-2} Z}. \qquad (18)$$

Since $lz + m$ can be expressed in the form

$$\lambda\{(c\beta^2 - a)z + 2b\beta^2\} + \mu(a - c\beta^2 + 2bz),$$

we see, from (17) and (18), that the integral (16) can be made to depend upon an algebraic expression and two integrals of the same form in which r is replaced by $r-1$ and $r-2$, respectively. By successive applications then of the formulæ of reduction, the integral (16) is made to depend upon two integrals of the form

$$\int\frac{(\lambda+\mu z)\,dz}{(z^2+\beta^2)^2 Z}, \quad \int\frac{(\lambda'+\mu' z)\,dz}{(z^2+\beta^2)Z}.$$

But putting $r = 2$ in (17) and (18), we see that the first of these integrals can be made to depend upon the second; and the latter integral, as has been remarked already, is evaluated in Art. 16.

If $Z = z^2 + \beta^2$, the integral (16) becomes

$$\int\frac{(lz+m)\,dz}{(z^2+\beta^2)^{r+\frac{1}{2}}},$$

which is equal to

$$\frac{-l}{(2r-1)(z^2+\beta^2)^{r-\frac{1}{2}}} + m\int\frac{dz}{(z^2+\beta^2)^{r+\frac{1}{2}}}.$$

For the latter integral we may investigate a formula of reduction as follows:—Let

$$P = \frac{z}{(z^2+\beta^2)^{r-\frac{1}{2}}};$$

then
$$\frac{dP}{dz} = \frac{1}{(z^2+\beta^2)^{r-\frac{1}{2}}} - \frac{(2r-1)z^2}{(z^2+\beta^2)^{r+\frac{1}{2}}}$$

$$= \frac{(2r-1)\beta^2}{(z^2+\beta^2)^{r+\frac{1}{2}}} - \frac{(2r-2)}{(z^2+\beta^2)^{r-\frac{1}{2}}}.$$

Hence we get

$$\int \frac{dz}{(z^2+\beta^2)^{r+\frac{1}{2}}} = \frac{z}{(2r-1)\beta^2 (z^2+\beta^2)^{r-\frac{1}{2}}} + \frac{2r-2}{(2r-1)\beta^2} \int \frac{dz}{(z^2+\beta^2)^{r-\frac{1}{2}}}, \quad (19)$$

by successive applications of which the given integral is ultimately made to depend upon

$$\int \frac{dz}{(z^2+\beta^2)^{\frac{3}{2}}} = \frac{z}{\beta^2 (z^2+\beta^2)^{\frac{1}{2}}}.$$

This integral is, however, more easily obtained by putting

$$z^2 + \beta^2 = y^2 z^2,$$

when we get

$$\int \frac{dz}{(z^2+\beta^2)^{r+\frac{1}{2}}} = \frac{-1}{\beta^{2r}} \int (y^2-1)^{r-1} \frac{dy}{y^{2r}},$$

which may be integrated at once by expanding $(y^2-1)^{r-1}$ by the Binomial Theorem.

51. In the preceding Articles there are included several integrals which seem to deserve special consideration.

For instance, under the form (13) is included the integral

$$\int \frac{x^m dx}{\sqrt{(a^2-x^2)}}, \quad (20)$$

the formula of reduction for which is, from (14),

$$\int \frac{x^m dx}{\sqrt{(a^2-x^2)}} = -\frac{x^{m-1}}{m}\sqrt{(a^2-x^2)} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2} dx}{\sqrt{(a^2-x^2)}}. \quad (21)$$

Changing then m into $m-2$, $m-4$, &c., successively, we come at last either to

$$\int \frac{x^2 dx}{\sqrt{(a^2-x^2)}} = -\tfrac{1}{2}x\sqrt{(a^2-x^2)} + \frac{a^2}{2}\int \frac{dx}{\sqrt{(a^2-x^2)}}$$

$$= -\tfrac{1}{2}x\sqrt{(a^2-x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a},$$

or to $$\int \frac{x\,dx}{\sqrt{(a^2-x^2)}} = -\sqrt{(a^2-x^2)};$$

so that, making use of these results, we find, when m is even,

$$\int \frac{x^m\,dx}{\sqrt{(a^2-x^2)}} = -\sqrt{(a^2-x^2)}\left\{\frac{x^{m-1}}{m} + \frac{(m-1)\,a^2x^{m-3}}{m\,(m-2)}\right.$$
$$\left. + \frac{(m-1)\,(m-3)}{m\,(m-2)\,(m-4)}\,a^4x^{m-5} + \ldots + \frac{(m-1)(m-3)\ldots 1}{m\,(m-2)(m-4)\ldots 2}\,a^{m-2}x\right\}$$
$$+ \frac{(m-1)\,(m-3)\ldots 1}{m\,(m-2)\ldots 2}\,a^m \sin^{-1}\frac{x}{a}; \qquad (22)$$

and if m is odd,

$$\int \frac{x^m\,dx}{\sqrt{(a^2-x^2)}} = -\sqrt{(a^2-x^2)}\left\{\frac{x^{m-1}}{m} + \frac{m-1}{m\,(m-2)}\,a^2x^{m-3}\right.$$
$$\left. + \frac{(m-1)(m-3)}{m\,(m-2)(m-4)}\,a^4x^{m-5} + \ldots + \frac{(m-1)(m-3)\ldots 2}{m\,(m-2)(m-4)\ldots 3}\,a^{m-1}\right\}. \qquad (23)$$

The latter integral may, however, be more easily evaluated by putting $x^2 = a^2 - y^2$, when it becomes

$$\int -\,(a^2-y^2)^{\frac{m-1}{2}}\,dy,$$

which may be obtained by expanding $(a^2-y^2)^{\frac{m-1}{2}}$ in a finite number of terms by the Binomial Theorem.

52. Again, let us consider the integral

$$\int \frac{x^m\,dx}{\sqrt{(2ax-x^2)}}. \qquad (24)$$

The formula of reduction in this case is from (13),

$$\int \frac{x^m\,dx}{\sqrt{(2ax-x^2)}} = \frac{-x^{m-1}\sqrt{(2ax-x^2)}}{m}$$
$$+ \frac{(2m-1)\,a}{m}\int \frac{x^{m-1}\,dx}{\sqrt{(2ax-x^2)}}, \qquad (25)$$

by means of which the value of the integral can be written down in the same way as in the preceding Article.

53. To find a formula of reduction for the integral

$$\int \frac{dx}{x^m \sqrt{(a+bx^2)}},$$

we take
$$P = \frac{\sqrt{(a+bx^2)}}{x^{m-1}}.$$

We have then

$$\frac{dP}{dx} = \frac{-(m-1)\sqrt{(a+bx^2)}}{x^{m-2}} + \frac{b}{x^{m-2}} \frac{1}{\sqrt{(a+bx^2)}}$$

$$= \frac{-(m-1)(a+bx^2)+bx^2}{x^{m-2}\sqrt{(a+bx^2)}}$$

$$= \frac{-(m-1)a}{x^m\sqrt{(a+bx^2)}} - \frac{(m-2)b}{x^{m-2}\sqrt{(a+bx^2)}}.$$

Hence, integrating, we get

$$\int \frac{dx}{x^m\sqrt{(a+bx^2)}} = \frac{-\sqrt{(a+bx^2)}}{(m-1)ax^{m-1}}$$

$$- \frac{(m-2)}{(m-1)}\frac{b}{a}\int \frac{dx}{x^{m-2}\sqrt{(a+bx^2)}}. \quad (26)$$

54. It often happens that an integral may be more easily treated at once by a special formula of reduction, instead of being reduced by the method adopted in Art. 48. For instance, the integral

$$\int (a^2 - x^2)^{n+\frac{1}{2}} dx, \quad (27)$$

where n is a positive integer, may be most easily treated as follows. Integrating by parts, we have

$$\int (a^2-x^2)^{n+\frac{1}{2}}dx = x(a^2-x^2)^{n+\frac{1}{2}} + (2n+1)\int x^2(a^2-x^2)^{n-\frac{1}{2}}dx$$

$$= x(a^2-x^2)^{n+\frac{1}{2}} - (2n+1)\int (a^2-x^2-a^2)(a^2-x^2)^{n-\frac{1}{2}}dx$$

$$= x(a^2-x^2)^{n+\frac{1}{2}} - (2n+1)\int (a^2-x^2)^{n+\frac{1}{2}}dx + (2n+1)a^2 \times \int (a^2-x^2)^{n-\frac{1}{2}}dx, \quad (28)$$

from which we get

$$\int (a^2-x^2)^{n+\frac{1}{2}}dx = \frac{x(a^2-x^2)^{n+\frac{1}{2}}}{2n+2} + \frac{(2n+1)a^2}{2n+2}\int (a^2-x^2)^{n-\frac{1}{2}}dx,$$

by which formula of reduction the given integral is ultimately made to depend upon

$$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}.$$

55. More generally, let us consider the integral

$$\int x^m(a+bx^2)^{n+\frac{1}{2}}dx. \quad (29)$$

Taking $P = x^{m-1}(a+bx^2)^{n+\frac{3}{2}}$, we have

$$\frac{dP}{dx} = (m-1)x^{m-2}(a+bx^2)^{n+\frac{3}{2}} + (2n+3)bx^m(a+bx^2)^{n+\frac{1}{2}}$$

$$= (a+bx^2)^{n+\frac{1}{2}}\{(m-1)ax^{m-2} + (m+2n+2)bx^m\}.$$

Hence, by integration, we obtain

$$\int x^m(a+bx^2)^{n+\frac{1}{2}}dx = \frac{x^{m-1}(a+bx^2)^{n+\frac{3}{2}}}{(m+2n+2)b} - \frac{(m-1)a}{(m+2n+2)b}\int x^{m-2}(a+bx^2)^{n+\frac{1}{2}}dx. \quad (30)$$

By this formula, when m is even, the given integral is ultimately made to depend upon $\int (a + bx^2)^{n+\frac{1}{2}} dx$, which may be treated in the same manner as the integral in the preceding Article. When m is odd, the last integral of the series is $\int x (a + bx^2)^{n+\frac{1}{2}} dx$, which is equal to

$$\frac{(a + bx^2)^{n+\frac{3}{2}}}{(2n + 3) b}.$$

56. In the case of the integral

$$\int (a + bx^2)^{n+\frac{1}{2}} \frac{dx}{x^m}, \tag{31}$$

we can find a formula of reduction in which the connected integral has both the indices diminished. We have

$$\int (a + bx^2)^{n+\frac{1}{2}} \frac{dx}{x^m} = \int \frac{-(a + bx^2)^{n+\frac{1}{2}}}{m - 1} d\left(\frac{1}{x^{m-1}}\right)$$

$$= \frac{-(a + bx^2)^{n+\frac{1}{2}}}{(m-1) x^{m-1}} + \frac{(2n + 1) b}{m - 1} \int (a + bx^2)^{n-\frac{1}{2}} \frac{dx}{x^{m-2}} \tag{32}$$

by integration by parts.

EXAMPLES.

1. $$\int \frac{x^3 dx}{\sqrt{(a + 2bx + cx^2)}} = \left\{\frac{x^2}{3c} - \frac{5bx}{6c^2} + \frac{15b^2 - 4ac}{6c^3}\right\} \sqrt{(a + 2bx + cx^2)}$$
$$+ \frac{(9abc - 15b^3)}{6c^3} \int \frac{dx}{\sqrt{(a + 2bx + cx^2)}}.$$

2. $$\int \frac{dx}{(x^2 - a^2)\sqrt{(1 - x^2)}} = \frac{1}{a\sqrt{(a^2 - 1)}} \tan^{-1} \left\{\frac{x\sqrt{(a^2 - 1)}}{a\sqrt{(1 - x^2)}}\right\},$$

or $$\frac{1}{2a\sqrt{(1 - a^2)}} \log \left\{\frac{a\sqrt{(1 - x^2)} + x\sqrt{(1 - a^2)}}{a\sqrt{(1 - x^2)} - x\sqrt{(1 - a^2)}}\right\}.$$

3. $$\int \frac{dx}{(x+a)^2\sqrt{(1+x^2)}} = \frac{a}{(1+a^2)^{\frac{3}{2}}} \log\left\{\frac{x+a}{1-ax+\sqrt{\{(1+a^2)(1+x^2)\}}}\right\}$$
$$-\frac{1}{\sqrt{(1+a^2)}}\,\frac{1}{x+a}+\frac{1}{(1+a^2)}\,\frac{a\sqrt{(1+x^2)}-x\sqrt{(1+a^2)}}{1-ax+\sqrt{\{(1+a^2)(1+x^2)\}}}.$$

4. $$\int \frac{dx}{(x+a)^2\sqrt{(1-x^2)}} = \frac{1}{a^2-1}\,\frac{\sqrt{(1-x^2)}}{x+a}$$
$$-\frac{2a}{(a^2-1)^{\frac{3}{2}}}\tan^{-1}\left\{\sqrt{\frac{(a-1)(1-x)}{(a+1)(1+x)}}\right\}.$$

5. $$\int \frac{(6bcz+ac-3b^2)\,dz}{(z^2+\beta^2)^2\sqrt{(a+2bz+cz^2)}} = \frac{2c^2(cz-b)\sqrt{(a+2bz+cz^2)}}{(ac-b^2)(z^2+\beta^2)},$$

where $$ac-3b^2=2c^2\beta^2.$$

6. $$\int \frac{dx}{(x^2+a)^2\sqrt{(1-x^2)}} = \frac{2a+1}{2(a^2+a)^{\frac{3}{2}}}\tan^{-1}\left(\frac{x\sqrt{(1+a)}}{\sqrt{\{a(1-x^2)\}}}\right)$$
$$+\frac{1}{(a^2+a)}\,\frac{x\sqrt{(1-x^2)}}{x^2+a}.$$

7. $$\int \frac{dx}{(x^2+a^2)^{\frac{5}{2}}} = \frac{x(2x^2+3a^2)}{3a^4(x^2+a^2)^{\frac{3}{2}}}.$$

8. $$\int \frac{x^4\,dx}{\sqrt{(a^2-x^2)}} = \frac{3a^4}{8}\sin^{-1}\frac{x}{a}-\frac{x}{8}(3a^2+2x^2)\sqrt{(a^2-x^2)}.$$

9. $$\int \frac{x^5\,dx}{\sqrt{(a^2-x^2)}} = -\frac{1}{15}(3x^4+4a^2x^2+8a^4)\sqrt{(a^2-x^2)}.$$

10. $$\int \frac{x^2\,dx}{\sqrt{(2ax-x^2)}} = -\tfrac{1}{2}(x+3a)\sqrt{(2ax-x^2)}+3a^2\sin^{-1}\left(\sqrt{\frac{x}{2a}}\right).$$

11. $$\int \frac{x^3\,dx}{\sqrt{(2ax-x^2)}} = 5a^3\sin^{-1}\left(\sqrt{\frac{x}{2a}}\right)-\frac{1}{6}(2x^2+5ax+15a^2)\sqrt{(2ax-x^2)}.$$

12. $$\int \frac{dx}{x^3\sqrt{(a^2-x^2)}} = \frac{1}{2a^3}\log\left\{\frac{a-\sqrt{(a^2-x^2)}}{x}\right\}-\frac{\sqrt{(a^2-x^2)}}{2a^2x^2}.$$

13. $$\int \frac{dx}{x^5\sqrt{(1+x^2)}} = \frac{(3x^2-2)}{8x^4}\sqrt{(1+x^2)}-\frac{3}{8}\log\left\{\frac{1+\sqrt{(1+x^2)}}{x}\right\}.$$

14. $\int (a^2 - x^2)^{\frac{5}{2}}\, dx = \frac{x}{48}(8x^4 - 26a^2x^2 + 33a^4)\sqrt{(a^2 - x^2)} + \frac{5a^6}{16}\sin^{-1}\frac{x}{a}.$

15. $$\int x^2 (a + bx^2)^{\frac{3}{2}}\, dx = \frac{x}{48b}(8b^2x^4 + 14abx^2 + 9a^2)\sqrt{(a + bx^2)}$$
$$+ \frac{a^3}{16b}\int \frac{dx}{\sqrt{(a + bx^2)}}.$$

16. $\int (a + bx^2)^{\frac{3}{2}}\frac{dx}{x^3} = \frac{(2bx^2 - a)}{2x^2}\sqrt{(a + bx^2)} + \frac{3ab}{2}\int \frac{dx}{x\sqrt{(a + bx^2)}}.$

57. As has been remarked in Art. 44, the method of treating integrals by formulae of reduction is not only applicable to those already considered, but is also of use in reducing expressions involving radicals of higher powers of the variable to certain fundamental standard forms. For instance, let us consider the integral

$$\int x^{m-1}(a + bx^2 + cx^4)^{p-1}\, dx, \tag{33}$$

where m is an integer and p is some fraction positive or negative. Putting $a + bx^2 + cx^4 = X$, we have

$$\frac{d(x^r X^p)}{dx} = x^{r-1}X^{p-1}\{rX + 2px(bx + 2cx^3)\}$$
$$= rax^{r-1}X^{p-1} + (r + 2p)\,bx^{r+1}X^{p-1} + (r + 4p)\,cx^{r+3}X^{p-1}.$$

By integration, then, we have an integral of the form (33) connected with two others in which the indices of x are increased or diminished by two and four, respectively. Thus, for example, taking $r = 1$, $p = \frac{1}{2}$, we get

$$\int \frac{x^4\,dx}{\sqrt{X}} = \frac{x\sqrt{X}}{3c} - \frac{2b}{3c}\int \frac{x^2\,dx}{\sqrt{X}} - \frac{a}{3c}\int \frac{dx}{\sqrt{X}}; \tag{34}$$

and the latter two integrals, as will be seen in the chapter on Elliptic Integrals, do not admit of any further reduction.

58. We now proceed to consider the investigation of formulae of reduction for integrals involving circular functions. Most of these can be transformed into the integrals of algebraic expressions already considered, but they seem to deserve separate treatment, on account of the simplicity of their forms and the frequency of their occurrence in the applications of the Integral Calculus.

Considering the integral $\int (\sin\theta)^n d\theta$,

we have
$$\int (\sin\theta)^n d\theta = -\int (\sin\theta)^{n-1} d(\cos\theta)$$
$$= -(\sin\theta)^{n-1}\cos\theta + (n-1)\int (\sin\theta)^{n-2}\cos^2\theta\, d\theta,$$

by the formula for integration by parts.

Hence
$$\int (\sin\theta)^n d\theta = -(\sin\theta)^{n-1}\cos\theta + (n-1)\int(\sin\theta)^{n-2}d\theta - (n-1)\int(\sin\theta)^n d\theta;$$

therefore
$$\int (\sin\theta)^n d\theta = -\frac{(\sin\theta)^{n-1}\cos\theta}{n} + \frac{(n-1)}{n}\int(\sin\theta)^{n-2}d\theta. \quad (35)$$

By successive applications of this formula we get, when n is even,
$$\int(\sin\theta)^n d\theta = -\frac{\cos\theta}{n}\left\{(\sin\theta)^{n-1} + \frac{(n-1)}{(n-2)}(\sin\theta)^{n-3} + \frac{(n-1)(n-3)}{(n-2)(n-4)}(\sin\theta)^{n-5} + \ldots + \frac{(n-1)(n-3)\ldots 1}{(n-2)(n-4)\ldots 2}\sin\theta\right\}$$
$$+ \frac{(n-1)(n-3)\ldots 1}{n(n-2)(n-4)\ldots 2}\,\theta; \quad (36)$$

and when n is odd,
$$\int(\sin\theta)^n d\theta = -\frac{\cos\theta}{n}\left\{\sin\theta)^{n-1} + \frac{(n-1)}{(n-2)}(\sin\theta)^{n-3} + \frac{(n-1)(n-3)}{(n-2)(n-4)}(\sin\theta)^{n-5} + \ldots + \frac{(n-1)(n-3)\ldots 2}{(n-2)(n-4)\ldots 1}\right\}. \quad (37)$$

In the latter case, however, the integral may be obtained by writing it in the form

$$-\int (1-\cos^2\theta)^{\frac{n-1}{2}} d(\cos\theta),$$

as $(1-\cos^2\theta)^{\frac{n-1}{2}}$ can be expanded in a finite number of terms.

In the same way as that made use of above, we get

$$\int \cos\theta)^n d\theta = \frac{\sin\theta(\cos\theta)^{n-1}}{n} + \frac{(n-1)}{n}\int(\cos\theta)^{n-2}d\theta; \quad (38)$$

and by means of this result we can write down expressions similar to (36) and (37).

59. Altering n into $n+2$, and then changing its sign in the formulæ of the preceding Article, we obtain, after transposition,

$$\int \frac{d\theta}{(\sin\theta)^n} = -\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}} + \frac{(n-2)}{(n-1)}\int\frac{d\theta}{(\sin\theta)^{n-2}}. \quad (39)$$

$$\int \frac{d\theta}{(\cos\theta)^n} = \frac{\sin\theta}{(n-1)(\cos\theta)^{n-1}} + \frac{(n-2)}{(n-1)}\int\frac{d\theta}{(\cos\theta)^{n-2}}. \quad (40)$$

By successive applications of (39) we find, when n is odd,

$$\int\frac{d\theta}{(\sin\theta)^n} = -\frac{\cos\theta}{(n-1)(\sin\theta)^{n-1}}\left\{1+\frac{(n-2)}{(n-3)}\sin^2\theta + \frac{(n-2)(n-4)}{(n-3)(n-5)}\sin^4\theta\right.$$

$$\left. + \dots + \frac{(n-2)(n-4)\dots 1}{(n-3)(n-5)\dots 2}(\sin\theta)^{n-3}\right\}$$

$$+\frac{(n-2)(n-4)\dots 1}{(n-1)(n-3)\dots 2}\log\tan\frac{\theta}{2}, \quad (41)$$

recollecting that

$$\int\frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2}.$$

When n is even, the integral may be obtained by writing it in the form

$$-\int (1+\cot^2\theta)^{\frac{n-2}{2}} d(\cot\theta),$$

and expanding

$$(1+\cot^2\theta)^{\frac{n-2}{2}}$$

in a finite number of terms.

In the same way, from (40), we can write down the value of $\int d\theta/(\cos\theta)^n$, when n is odd; and when n is even this integral can be obtained by writing it in the form

$$\int (1+\tan^2\theta)^{\frac{n-2}{2}} d(\tan\theta).$$

60. It may be observed that the integrals of Art. 58 can be obtained very readily by expressing the powers of $\sin\theta$ and $\cos\theta$ in terms of the simple dimensions of the sines or cosines of multiple angles. Thus, from trigonometrical considerations, we have

$$2^{n-1}(\cos\theta)^n = \cos n\theta + n\cos(n-2)\theta + \frac{n\,.\,n-1}{1\,.\,2}\cos(n-4)\theta + \&c.$$

Hence we get

$$\int (\cos\theta)^n d\theta = \frac{1}{2^{n-1}}\left\{\frac{\sin n\theta}{n} + \frac{n\sin(n-2)\theta}{n-2} + \frac{n\,.\,n-1}{1\,.\,2}\frac{\sin(n-4)\theta}{n-4} + \&c.\right\}, \qquad (42)$$

in which the last term is, if $n=2r$,

$$\frac{1}{2^r}\,\frac{1\,.\,3\,.\,5\,\ldots\,2r-1}{1\,.\,2\,.\,3\,\ldots\,r}\,\theta,$$

and if $n = 2r - 1$,

$$\frac{1}{2^{2r-2}} \frac{(2r-1)(2r-2)\ldots(2\,.\,1)}{1\,.\,2\,.\,3\ldots r-1} \sin\theta.$$

In the same way we can evaluate the integral $\int(\sin\theta)^n d\theta$ by expanding $(\sin\theta)^n$ in a series of sines or cosines of multiple angles, according as n is odd or even.

Examples.

1. $\int \sin^4\theta\, d\theta = -\frac{1}{8}\sin\theta\cos\theta\,(3 + 2\sin^2\theta) + \frac{3}{8}\theta.$

2. $\int \cos^6\theta\, d\theta = \frac{1}{48}\sin\theta\cos\theta\,(8\cos^4\theta + 10\cos^2\theta + 15) + \frac{5}{16}\theta.$

3. $\int \frac{d\theta}{\sin^3\theta} = -\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{2}\log\tan\frac{\theta}{2}.$

4. $\int \frac{d\theta}{\cos^5\theta} = \frac{\sin\theta}{8\cos^4\theta}(2 + 3\cos^2\theta) + \frac{3}{8}\log(\sec\theta + \tan\theta).$

5. $\int \frac{d\theta}{\sin^6\theta} = -\frac{1}{15}\cot\theta\,(3\cot^4\theta + 10\cot^2\theta + 15).$

6. $\int \cos^4\theta\, d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta.$

7. $\int \cos^5\theta\, d\theta = \frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin\theta.$

61. We now proceed to consider the integral

$$\int(\sin\theta)^n(\cos\theta)^m d\theta. \qquad (43)$$

Before, however, investigating a formula of reduction, we notice some cases in which the integral may be obtained more

simply otherwise. First, if either of the numbers m or n is an odd positive integer, say, $m = 2r + 1$, we have

$$\int (\sin \theta)^n (\cos \theta)^m d\theta = \int (\sin \theta)^n (1 - \sin^2 \theta)^r d(\sin \theta),$$

which may be integrated at once by expending $(1 - \sin^2 \theta)^r$.

Similarly, if $n = 2r + 1$, we have

$$\int (\sin \theta)^n (\cos \theta)^m d\theta = - \int (1 - \cos^2 \theta)^r (\cos \theta)^m d(\cos \theta).$$

Again, if the sum of m and n is a negative even integer, say, $m + n = -2r$, we have

$$\int (\sin \theta)^n (\cos \theta)^m d\theta = \int (\tan \theta)^n (\sec \theta)^{2r} d\theta$$

$$= \int (\tan \theta)^n (1 + \tan^2 \theta)^{r-1} d(\tan \theta),$$

or $$\int (\sin \theta)^n (\cos \theta)^m d\theta = \int (\operatorname{cosec} \theta)^{2r} (\cot \theta)^m d\theta$$

$$= - \int (1 + \cot^2 \theta)^{r-1} (\cot \theta)^m d(\cot \theta).$$

62. If the integral considered in the preceding Article does not come under the special cases mentioned there, we must make use of a formula of reduction which may be obtained as follows:—We have

$$\int (\sin \theta)^n (\cos \theta)^m d\theta = - \frac{1}{m+1} \int (\sin \theta)^{n-1} d(\cos^{m+1} \theta)$$

$$= - \frac{(\sin \theta)^{n-1} (\cos \theta)^{m+1}}{m+1} + \frac{(n-1)}{m+1} \int (\cos \theta)^{m+2} (\sin \theta)^{n-2} d\theta, \quad (44)$$

by the formula for integration by parts.

Hence, $$\int (\sin \theta)^n (\cos \theta)^m d\theta = - \frac{(\sin \theta)^{n-1} (\cos \theta)^{m+1}}{m+1}$$

$$+ \frac{n-1}{m+1} \int (\cos \theta)^{n-2} (\cos \theta)^m d\theta - \frac{(n-1)}{m+1} \int (\cos \theta)^m (\sin \theta)^n d\theta;$$

therefore, bringing across the last term to the left-hand side and dividing by $(m+n)(m+1)$, we get

$$\int (\sin\theta)^n (\cos\theta)^m d\theta = -\frac{(\sin\theta)^{n-1}(\cos\theta)^{m+1}}{m+n} + \frac{(n-1)}{m+n}\int (\sin\theta)^{n-2}(\cos\theta)^m d\theta. \qquad (45)$$

In the same way we obtain

$$\int (\sin\theta)^n (\cos\theta)^m d\theta = \frac{(\sin\theta)^{n+1}(\cos\theta)^{m-1}}{n+1} + \frac{m-1}{n+1}\int (\sin\theta)^{n+2}(\cos\theta)^{m-2} d\theta, \qquad (46)$$

and

$$\int (\sin\theta)^n (\cos\theta)^m d\theta = \frac{(\sin\theta)^{n+1}(\cos\theta)^{m-1}}{m+n} + \frac{(m-1)}{m+n}\int (\sin\theta)^{n}(\cos\theta)^{m-2} d\theta, \qquad (47)$$

By successive applications, then, of the latter formula we get, if m is supposed to be an even positive integer,

$$\int (\sin\theta)^n (\cos\theta)^m d\theta = \frac{(\sin\theta)^{n+1}}{m+n}\left\{(\cos\theta)^{m-1} + \frac{(m-1)}{(m+n-2)}(\cos\theta)^{m-3} + \frac{(m-1)(m-3)}{(m+n-2)(m+n-4)}(\cos\theta)^{m-5} + \&c.\right\}$$
$$+ \frac{(m-1)(m-3)\ldots 3.1}{(m+n)(m+n-2)\ldots(m+2)}\int (\sin\theta)^n d\theta; \qquad (48)$$

and the latter integral is known from (36) or (37), according as n is positive or negative.

In the same way, if n is an even positive integer, we find, from (47),

$$\int (\sin\theta)^n (\cos\theta)^m \, d\theta = \frac{-(\cos\theta)^{m+1}}{m+n} \left\{ (\sin\theta)^{n-1} \right.$$

$$\left. + \frac{(n-1)}{(m+n-2)} (\sin\theta)^{n-3} + \frac{(n-1)(n-3)}{(m+n-2)(m+n-4)} (\sin\theta)^{n-5} + \&c. \right\}$$

$$+ \frac{(n-1)(n-3)\ldots 3.1}{(m+n)(m+n-2)\ldots(m+2)} \int (\cos\theta)^m d\theta. \quad (49)$$

63. If one of the indices in (43) is positive and the other negative, we can obtain a formula of reduction in which both of these quantities in the connected integral are diminished. Altering the sign of m in (44), we have

$$\int \frac{(\sin\theta)^n d\theta}{(\cos\theta)^m} = \frac{(\sin\theta)^{n-1}}{(m-1)(\cos\theta)^{m-1}} - \frac{(n-1)}{m-1} \int \frac{(\sin\theta)^{n-2} d\theta}{(\cos\theta)^{m-2}}. \quad (50)$$

Since we may suppose n to be even, by this formula of reduction the given integral is ultimately made to depend upon $\int (\sin\theta)^{n-m} d\theta$, or

$$\int \frac{(\sin\theta)^{n-m+1} d\theta}{\cos\theta} = \int \frac{(\sin\theta)^{n-m+1} d\sin\theta}{1-\sin^2\theta}, \text{ if } m < n,$$

and

$$\int \frac{d\theta}{(\cos\theta)^{m-n}}, \text{ if } m > n.$$

Again, altering the sign of n in (46), we have

$$\int \frac{(\cos\theta)^m d\theta}{(\sin\theta)^n} = \frac{-(\cos\theta)^{m-1}}{(n-1)(\sin\theta)^{n-1}} - \frac{(m-1)}{n-1} \int \frac{(\cos\theta)^{m-2} d\theta}{(\sin\theta)^{n-2}}. \quad (51)$$

64. If both the indices in the integral which we have been considering are negative, we may find a formula of reduction as follows. Writing the integral in the form

$$\int \frac{d\theta}{(\sin\theta)^n(\cos\theta)^m}, \tag{52}$$

where $m + n$ is supposed to be odd, and taking

$$P = \frac{1}{(\sin\theta)^{n-1}(\cos\theta)^{m-1}},$$

we have

$$\frac{dP}{d\theta} = \frac{-(n-1)}{(\sin\theta)^n(\cos\theta)^{m-2}} + \frac{(m-1)(1-\cos^2\theta)}{(\sin\theta)^n(\cos\theta)^m}$$

$$= \frac{m-1}{(\sin\theta)^n(\cos\theta)^m} - \frac{(m+n-2)}{(\sin\theta)^n(\cos\theta)^{m-2}}.$$

Hence, by integration, we find

$$\int \frac{d\theta}{(\sin\theta)^n(\cos\theta)^m} = \frac{1}{(m-1)(\sin\theta)^{n-1}(\cos\theta)^{m-1}} + \frac{(m+n-2)}{m-1}\int \frac{d\theta}{(\sin\theta)^n(\cos\theta)^{m-2}}. \tag{53}$$

In the same way we obtain

$$\int \frac{d\theta}{(\sin\theta)^n(\cos\theta)^m} = \frac{1}{(n-1)(\sin\theta)^{n-1}(\cos\theta)^{m-1}} + \frac{(m+n-2)}{n-1}\int \frac{d\theta}{(\sin\theta)^{n-2}(\cos\theta)^m}. \tag{54}$$

By successive applications, then, of both these formulae we can ultimately make an integral of the form (52) depend upon one of the integrals, according as n is even or odd,

$$\int \frac{d\theta}{\cos\theta} = \log(\sec\theta + \tan\theta), \quad \int \frac{d\theta}{\sin\theta} = \log\tan\frac{\theta}{2}.$$

65. We consider here the integrals $\int (\tan\theta)^n d\theta$, $\int (\cot\theta)^n d\theta$. Although these are included in the case treated in Art. 63, we give a separate investigation on account of the simplicity of the results. We have

$$\int (\tan\theta)^n d\theta = \int (\tan\theta)^{n-2}(\sec^2\theta - 1)\, d\theta$$

$$= \int (\tan\theta)^{n-2} d(\tan\theta) - \int (\tan\theta)^{n-2} d\theta,$$

or $$\int (\tan\theta)^n d\theta = \frac{(\tan\theta)^{n-1}}{n-1} - \int (\tan\theta)^{n-2} d\theta.$$

Hence, we get at once

$$\int (\tan\theta)^n d\theta = \frac{(\tan\theta)^{n-1}}{n-1} - \frac{(\tan\theta)^{n-3}}{n-3} + \frac{(\tan\theta)^{n-5}}{n-5} \dots \&c.; \quad (55)$$

the last term being $(-1)^{\frac{n}{2}}\theta$, if n is even; and $(-1)^{\frac{n+1}{2}}\log\cos\theta$, if n is odd.

In a similar manner we find

$$\int (\cot\theta)^n d\theta = -\frac{(\cot\theta)^{n-1}}{n-1} - \int (\cot\theta)^{n-2} d\theta. \quad (56)$$

Examples.

1. $\displaystyle\int \frac{\cos^3\theta\, d\theta}{\sin^2\theta} = -\frac{(1+\sin^2\theta)}{\sin\theta}$.

2. $\displaystyle\int \frac{\sin^5\theta\, d\theta}{\cos^2\theta} = \frac{1}{3}\sec\theta\,(3 + 6\cos^2\theta - \cos^4\theta)$.

3. $\displaystyle\int \sin^3\theta\cos^7\theta\, d\theta = \frac{1}{40}\cos^8\theta\,(4\cos^2\theta - 5)$.

4. $\displaystyle\int \frac{\cos^3\theta\, d\theta}{\sqrt{(\sin\theta)}} = 2\sqrt{(\sin\theta)}\left(1 - \frac{1}{5}\sin^2\theta\right)$.

5. $\displaystyle\int \sqrt{(\cos\theta)}\sin^5\theta\, d\theta = 2(\cos\theta)^{\frac{3}{2}}\left\{\frac{2}{7}\cos^2\theta - \frac{1}{11}\cos^4\theta - \frac{1}{3}\right\}$.

6. $\int \sin^2\theta \cos^5\theta\, d\theta = \sin^3\theta \left(\frac{1}{7}\sin^4\theta - \frac{2}{5}\sin^2\theta + \frac{1}{3}\right).$

7. $\int \frac{d\theta}{\sin^6\theta \cos^4\theta} = 4\tan\theta + \frac{1}{3}\tan^3\theta - 6\cot\theta - \frac{4}{3}\cot^3\theta - \frac{1}{5}\cot^5\theta.$

8. $\int \frac{d\theta}{\sin\theta \cos^5\theta} = \frac{1}{4}\tan^4\theta + \tan^2\theta + \log\tan\theta.$

9. $\int \frac{d\theta}{(\sin\theta)^{\frac{1}{2}}(\cos\theta)^{\frac{7}{2}}} = 2(\tan\theta)^{\frac{1}{2}}\left(1 + \frac{1}{5}\tan^2\theta\right).$

10. $\int \frac{\cos^3\theta\, d\theta}{\sin\theta} = -\frac{1}{2}\sin^2\theta + \log\sin\theta.$

11. $\int \cos^4\theta \sin^2\theta\, d\theta = \frac{1}{48}\sin\theta\cos\theta(3 + 2\cos^2\theta - 8\cos^4\theta) + \frac{\theta}{16}.$

12. $\int \sin^4\theta \cos^4\theta\, d\theta = \frac{3}{128}\theta - \frac{1}{128}\sin\theta\cos\theta\cos 2\theta(3 + 8\sin^2\theta\cos^2\theta).$

13. $\int \sin^4\theta \cos^6\theta\, d\theta = \frac{3}{256}\theta - \frac{1}{256}\sin\theta\cos\theta\cos 2\theta(3 + 8\sin^2\theta\cos^2\theta)$

$$+ \frac{1}{10}\sin^5\theta\cos^5\theta.$$

14. $\int \frac{d\theta}{\sin\theta \cos^4\theta} = \frac{1 + 3\cos^2\theta}{3\cos^3\theta} + \log\tan\frac{\theta}{2}.$

15. $\int \frac{d\theta}{\sin^3\theta \cos^2\theta} = \frac{2 - 3\cos^2\theta}{2\sin^2\theta\cos\theta} + \frac{3}{2}\log\tan\frac{\theta}{2}.$

16. $\int \frac{d\theta}{\sin^5\theta \cos^2\theta} = \frac{8 - 25\cos^2\theta + 15\cos^4\theta}{8\sin^4\theta\cos\theta} + \frac{15}{8}\log\tan\frac{\theta}{2}.$

17. $\int \tan^4\theta\, d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta.$

18. $\int (\cot\theta)^5\, d\theta = -\frac{1}{4}\cot^4\theta + \frac{1}{2}\cot^2\theta + \log\sin\theta.$

19. $\int (\cot\theta)^6\, d\theta = -\frac{1}{5}\cot^5\theta + \frac{1}{3}\cot^3\theta - \cot\theta - \theta.$

66. To find a formula of reduction for the integral

$$\int \frac{d\theta}{(a + b\cos\theta)^n}, \tag{57}$$

where n is a positive integer, we take

$$P = \frac{\sin\theta}{\Theta^{n-1}},$$

where $$a + b\cos\theta = \Theta.$$

We have then, by differentiation,

$$\frac{dP}{d\theta} = \frac{\cos\theta}{\Theta^{n-1}} + \frac{(n-1)\,b\sin^2\theta}{\Theta^n}$$

$$= \frac{\cos\theta}{\Theta^{n-1}} + \frac{(n-1)\,b}{\Theta^n} - \frac{(n-1)\,b\cos^2\theta}{\Theta^n}.$$

Hence, putting $(\Theta - a)/b$ for $\cos\theta$, we get

$$\frac{dP}{d\theta} = \frac{\Theta - a}{b\Theta^{n-1}} + \frac{(n-1)\,b}{\Theta^n} - \frac{(n-1)(\Theta^2 - 2a\Theta + a^2)}{b\Theta^n}$$

$$= -\frac{(n-2)}{b\Theta^{n-2}} + \frac{(2n-3)\,a}{b\Theta^{n-1}} - \frac{(n-1)(a^2-b^2)}{b\Theta^n}.$$

Therefore, integrating and transposing, we get

$$\int \frac{d\theta}{\Theta^n} = \frac{-b\sin\theta}{(n-1)(a^2-b^2)\,\Theta^{n-1}} + \frac{(2n-3)\,a}{(n-1)(a^2-b^2)}\int\frac{d\theta}{\Theta^{n-1}}$$

$$- \frac{(n-2)}{(n-1)(a^2-b^2)}\int\frac{d\theta}{\Theta^{n-2}}. \tag{58}$$

By successive applications of this formula the given integral is ultimately made to depend upon the integrals

$$\int\frac{d\theta}{(a+b\cos\theta)^2}, \quad \int\frac{d\theta}{a+b\cos\theta}.$$

But, taking $n = 2$ in (58), we have the first of these expressed in terms of the second; and the latter integral is evaluated in Art. 19.

67. The integral

$$\int \frac{d\phi}{(a^2 \cos^2\phi + b^2 \sin^2\phi)^n} \tag{59}$$

evidently comes under the form considered in the preceding Article by putting $2\phi = \theta$; but we consider it worth while mentioning here a substitution by which (59) assumes a form admitting of a more immediate integration. Taking

$$\tan\phi = \frac{a}{b}\tan\theta,$$

we have $\quad d\phi = d\tan^{-1}\left(\frac{a}{b}\tan\theta\right) = \dfrac{ab\,d\theta}{a^2\sin^2\theta + b^2\cos^2\theta};$

also $\quad a^2\cos^2\phi + b^2\sin^2\phi = \dfrac{a^2b^2}{a^2\sin^2\theta + b^2\cos\theta};$

so that we have

$$\int \frac{d\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^n} = \frac{1}{(ab)^{2n-1}}\int (a^2\sin^2\theta + b^2\cos^2\theta)^{n-1}\,d\theta; \tag{60}$$

but the latter integral can be evaluated at once by expanding

$$(a^2\sin^2\theta + b^2\cos^2\theta)^{n-1}, \text{ or } \{a^2 + (b^2 - a^2)\cos^2\theta\}^{n-1}$$

in a finite number of terms, and integrating each term separately by Art. 58.

68. We proceed now to consider the integral

$$\int x^n \cos ax\,dx.$$

Integrating by parts, we have

$$\int x^n \cos ax\,dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx,$$

and $$\int x^{n-1} \sin ax dx = -\frac{x^{n-1}\cos ax}{a} + \frac{(n-1)}{a}\int x^{n-2}\cos ax dx;$$

so that, eliminating $\int x^{n-1} \sin ax dx$, we get

$$\int x^n \cos ax dx = \frac{x^{n-1}}{a^2}(n\cos ax + ax \sin ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\cos ax dx. \tag{61}$$

By means of this formula of reduction the proposed integral is ultimately made to depend upon

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad \text{or} \quad \int x \cos ax dx = \frac{x}{a}\sin ax + \frac{n}{a^2}\cos ax,$$

according as n is even or odd.

In the same way we find

$$\int x^n \sin ax dx = \frac{x^{n-1}}{a^2}(n\sin ax - ax\cos ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax dx. \tag{62}$$

69. Altering n into $n+2$ in (61) and (62), and then changing the sign of n, we obtain, after transposing and dividing by $(n-1)(n-2)/a^2$,

$$\int \frac{\cos ax dx}{x^n} = \frac{ax \sin ax - (n-2)\cos ax}{(n-1)(n-2)x^{n-1}} - \frac{a^2}{(n-1)(n-2)}\int \frac{\cos ax dx}{x^{n-2}}, \tag{63}$$

$$\int \frac{\sin ax dx}{x^n} = -\frac{\{ax \cos ax + (n-2)\sin ax\}}{(n-1)(n-2)x^{n-1}} - \frac{a^2}{(n-1)(n-2)}\int \frac{\sin ax dx}{x^{n-2}}. \tag{64}$$

By means of these formulæ of reduction the integrals

$$\int \frac{\cos ax\, dx}{x^n}, \quad \int \frac{\sin ax\, dx}{x^n}$$

are ultimately made to depend upon

$$\int \frac{\cos ax\, dx}{x}, \quad \int \frac{\sin ax\, dx}{x},$$

which are irreducible, and can only be obtained in infinite series by expanding $\cos ax$ and $\sin ax$.

70. We can find a formula of reduction for the integral

$$\int e^{ax} (\sin x)^n dx$$

as follows :—Integrating by parts, we have

$$\int e^{ax} (\sin x)^n dx = \frac{e^{ax}}{a} (\sin x)^n - \frac{n}{a} \int e^{ax} (\sin x)^{n-1} \cos x\, dx;$$

but

$$\int e^{ax} (\sin x)^{n-1} \cos x\, dx = \frac{e^{ax}}{a} (\sin x)^{n-1} \cos x$$

$$- \frac{1}{a} \int e^{ax} \{(n-1)(\sin x)^{n-2} - n(\sin x)^n\} dx$$

$$= e^{ax} (\sin x)^{n-1} \cos x - \frac{n-1}{a} \int e^{ax} (\sin x)^{n-2} dx + \frac{n}{a} \int e^{ax} (\sin x)^n dx.$$

Hence, eliminating

$$\int e^{ax} (\sin x)^{n-1} \cos x\, dx,$$

and solving for

$$\int e^{ax} (\sin x)^n dx,$$

we get

$$\int e^{ax} (\sin x)^n dx = e^{ax} (\sin x)^{n-1} \frac{(a \sin x - n \cos x)}{n^2 + a^2}$$

$$+ \frac{n(n-1)}{n^2 + a^2} \int e^{ax} (\sin x)^{n-2} dx. \qquad (65)$$

In the same way we obtain

$$\int e^{ax}(\cos x)^n dx = e^{ax}(\cos x)^{n-1}\frac{(a\cos x + n\sin x)}{n^2+a^2}$$

$$+\frac{n(n-1)}{n^2+a^2}\int e^{ax}(\cos x)^{n-2}dx. \qquad (66)$$

By these formulae the proposed integrals are made to depend upon

$$\int e^{ax}dx = \frac{1}{a}e^{ax},$$

when n is even; and upon

$$\int e^{ax}\sin x\,dx, \quad \int e^{ax}\cos x\,dx,$$

respectively, when n is odd. The latter integrals have been already given in Art. 17, but may be found at once from (65) and (66) by putting $n = 1$.

71. We can investigate directly a formula of reduction for the integral

$$\int \cos ax\,(\sin x)^n dx,$$

and thus connect it with another of the same form in which the index of $\sin x$ is diminished by two. This formula may be, however, obtained at once from the preceding Article by the use of the imaginary.

Putting ia for a in (65), and equating the real parts on both sides of the resulting equation, we get

$$\int \cos ax\,(\sin x)^n dx = \frac{(\sin x)^{n-1}}{a^2-n^2}(n\cos x\cos ax + a\sin x\sin ax)$$

$$-\frac{n(n-1)}{a^2-n^2}\int \cos ax\,(\sin x)^{n-2}dx. \qquad (67)$$

Also, equating the imaginary parts, we obtain

$$\int \sin ax\,(\sin x)^n\,dx = (\sin x)^{n-1}\,\frac{(a\sin x\cos ax - n\cos x\sin ax)}{n^2-a^2} + \frac{n(n-1)}{n^2-a^2}\int \sin ax\,(\sin x)^{n-2}\,dx. \quad (68)$$

Again, from (66), in the same way we obtain the formulae of reduction

$$\int \cos ax\,(\cos x)^n\,dx = (\cos x)^{n-1}\,\frac{(n\sin x\cos ax - a\cos x\sin ax)}{n^2-a^2} + \frac{n(n-1)}{n^2-a^2}\int \cos ax\,(\cos x)^{n-2}\,dx, \quad (69)$$

$$\int \sin ax\,(\cos x)^n\,dx = (\cos x)^{n-1}\,\frac{(n\sin x\sin ax + a\cos x\cos ax)}{n^2-a^2} + \frac{n(n-1)}{n^2-a^2}\int \sin ax\,(\cos x)^{n-2}\,dx. \quad (70)$$

72. It may be observed, that if in any one of the preceding integrals a is an integer, we can obtain a formula of reduction connecting the given integral with another of a similar form, in which both n and a are diminished by unity. Thus, for example, integrating by parts, we have

$$\int \cos ax\,(\sin x)^n\,dx = \frac{1}{a}\sin ax\,(\sin x)^n - \frac{n}{a}\int \sin ax\cos x\,(\sin x)^{n-1}\,dx$$

$$= \frac{1}{a}\sin ax\,(\sin x)^n - \frac{n}{a}\int \{\sin(a-1)x + \sin x\cos ax\}\,(\sin x)^{n-1}\,dx$$

$$= \frac{1}{a} \sin ax\,(\sin x)^n - \frac{n}{a}\int \sin(a-1)x\,(\sin x)^{n-1}dx$$
$$- \frac{n}{a}\int \cos ax\,(\sin x)^n dx;$$

hence, solving for $\int \cos ax\,(\sin x)^n dx$, we get

$$\int \cos ax\,(\sin x)^n dx = \frac{\sin ax\,(\sin x)^n}{n+a}$$
$$- \frac{n}{n+a}\int \sin(a-1)x\,(\sin x)^{n-1}dx. \qquad (71)$$

By means of this formula of reduction, and a similar one for $\int \sin ax\,(\sin x)^n dx$, we can determine the value of the proposed integral. In exactly the same way we can obtain similar formulae of reduction for the three other integrals considered in the preceding Article.

73. More generally, substituting $a + ib$ for a in (65), and equating the real parts on both sides of the resulting equation, we obtain

$$\int e^{ax} \cos bx\,(\sin x)^n dx = \frac{e^{ax}}{c}(\sin x)^{n-1}\{(a\sin x - n\cos x)\cos(bx-\lambda)$$
$$- b\sin x \sin(bx-\lambda)\}$$
$$+ \frac{n(n-1)}{c}\int e^{ax}\cos(bx-\lambda)(\sin x)^{n-2}dx, \qquad (72)$$

where $n^2 + a^2 - b^2 = c\cos\lambda$, $2ab = c\sin\lambda$.

In the same way we can obtain formulae of reduction for the integrals $\int e^{ax}\sin bx\,(\sin x)^n dx$, $\int e^{ax}\cos bx\,(\cos x)^n dx$, $\int e^{ax}\sin bx\,(\cos x)^n dx$.

74. We may here investigate a formula of reduction for $\int x^n e^{ax} dx$, where n is a positive integer, although we have in effect already determined the value of this integral in Art. 18.

Integrating by parts, we have

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx. \tag{73}$$

By successive applications of this formula, we ultimately come to $\int e^{ax}\,dx = e^{ax}/a$, and thus arrive at the result (28) of Chap. I.

If n is of the form $(2m+1)/2$, the proposed integral can be made to depend upon

$$\int \frac{e^{ax}\,dx}{\sqrt{x}} = 2\int e^{ay^2}dy, \text{ if } x = y^2.$$

The latter integral can only be obtained by expanding e^{ay^2} in an infinite series, as follows:

$$\int e^{ay^2}\,dy = \int \left(1 + ay^2 + \frac{a^2y^4}{1.2} + \&c.\right)dy$$

$$= y + \frac{ay^3}{3} + \frac{1}{5}\,\frac{a^2y^4}{1.2} + \&c.$$

To find a formula of reduction for

$$\int \frac{e^{ax}\,dx}{x^n},$$

we have

$$\int \frac{e^{ax}\,dx}{x^n} = -\frac{1}{n-1}\int e^{ax} d\left(\frac{1}{x^{n-1}}\right)$$

$$= -\frac{e^{ax}}{(n-1)\,x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}\,dx}{x^{n-1}}. \tag{74}$$

By means of this formula the proposed integral is ultimately made to depend upon

$$\int \frac{e^{ax}\,dx}{x},$$

which can only be obtained by expanding e^{ax} in an infinite series. We find thus

$$\int \frac{e^{ax}\,dx}{x} = \log x + ax + \frac{a^2x^2}{1.2^2} + \frac{a^3x^3}{1.2.3^3} + \&c. \quad (75)$$

75. Putting $a + ib$ for a in (73), and equating the real and imaginary parts, respectively, on both sides of the resulting equation, we obtain

$$\int x^n e^{ax} \cos bx\,dx = x^n e^{ax}\,\frac{(a \cos bx + b \sin bx)}{a^2 + b^2}$$
$$- \frac{n}{a^2 + b^2} \int x^{n-1} e^{ax}\,(a \cos bx + b \sin bx)\,dx \quad (76)$$

$$\int x^n e^{ax} \sin bx\,dx = x^n e^{ax}\,\frac{(a \sin bx - b \cos bx)}{a^2 + b^2}$$
$$- \frac{n}{a^2 + b^2} \int x^{n-1} e^{ax}\,(a \sin bx - b \cos bx)\,dx, \quad (77)$$

which constitute formulae of reduction for the integrals

$$\int x^n e^{ax} \cos bx\,dx, \quad \int x^n e^{ax} \sin bx\,dx.$$

In exactly the same way we can obtain from (74) formulae of reduction for

$$\int e^{ax} \cos bx\,\frac{dx}{x^n}, \quad \int e^{ax} \sin bx\,\frac{dx}{x^n}.$$

76. The formula of reduction for $\int x^m (\log x)^n \, dx$ may be found from (73) by putting $x = e^z$. We obtain thus

$$\int x^m (\log x)^n \, dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx. \quad (78)$$

Similarly, from (74), we get

$$\int \frac{x^m \, dx}{(\log x)^n} = - \frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m \, dx}{(\log x)^{n-1}}. \quad (79)$$

EXAMPLES.

1. $$\int \frac{d\theta}{(a + b\cos\theta)^3} = - b \sin\theta \frac{(4a^2 - b^2 + 3ab\cos\theta)}{2c^4 (a + b\cos\theta)} + \frac{(2a^2 + b^2)}{c^5} \tan^{-1} \left\{ \frac{(a-b)}{c} \tan \tfrac{1}{2}\theta \right\},$$

where $a^2 - b^2 = c^2$.

2. $$\int \frac{d\theta}{(a^2 \cos^2\theta + b^2 \sin^2\theta)^2} = \frac{(a^2 + b^2)\phi}{2a^3 b^3} - \frac{(a^2 - b^2)\sin\phi \cos\phi}{4a^3 b^3},$$

where $\tan\phi = \frac{b}{a} \tan\theta$.

3. $$\int x^4 \cos ax \, dx = \frac{x \cos ax}{a^4} (4a^2x^2 - 24) + \frac{\sin ax}{a^5} (a^4x^4 - 12a^2x^2 + 24).$$

4. $$\int x^3 \sin x \, dx = (3x^2 - 6) \sin x - (x^3 - 6x) \cos x.$$

5. $$\int \frac{\cos x \, dx}{x^3} = \frac{x \sin x - \cos x}{2x^2} - \frac{1}{2} \int \frac{\cos x \, dx}{x}.$$

6. $$\int \frac{\sin x \, dx}{x^4} = \frac{(x^2 - 2)\sin x - x \cos x}{6x^3} - \frac{1}{6} \int \frac{\cos x \, dx}{x}.$$

7. $$\int e^{ax} \sin^3 x \, dx = \frac{e^{ax} \sin^2 x \, (a \sin x - 3\cos x)}{a^2 + 9} + \frac{6e^{ax} (a \sin x - \cos x)}{(a^2 + 9)(a^2 + 1)}.$$

8. $$\int e^{ax}\cos^2 x\,dx = \frac{e^{ax}}{a(a^2+4)}(a^2\cos^2 x + a\sin 2x + 2).$$

9. $$\int \cos 3x\sin^2 x\,dx = \frac{1}{5}\sin x(2\cos x\cos 3x + 3\sin x\sin 3x) - \frac{2}{15}\sin 3x.$$

10. $$\int \sin 3x\cos^2 x\,dx = -\frac{1}{5}\cos x(2\sin x\sin 3x + 3\cos x\cos 3x) + \frac{2}{15}\cos 3x.$$

11. $$\int \cos 4x\sin^2 x\,dx = \frac{1}{6}\sin 4x\sin^2 x - \frac{1}{24}(2\sin 2x - \cos 4x).$$

12. $$\int e^{ax}\cos bx\sin x\,dx = \frac{e^{ax}}{c}\{(a\sin x - \cos x)\cos(bx-\lambda) - b\sin x\sin(bx-\lambda)\},$$

where $$1 + a^2 - b^2 = c\cos\lambda, \quad 2ab = c\sin\lambda.$$

13. $$\int x^{\frac{5}{2}} e^x dx = x^{\frac{1}{2}} e^x\left(x^2 - \frac{5}{2}x + \frac{15}{4}\right) - \frac{15}{8}\int e^x\frac{dx}{\sqrt{x}}.$$

14. $$\int e^{ax}\frac{dx}{x^3} = -\frac{e^{ax}}{2x^2}(1+ax) + \frac{1}{2}a^2\int e^{ax}\frac{dx}{x}.$$

15. $$\int xe^{ax}\cos bx\,dx = \frac{xe^{ax}}{2(a^2+b^2)}(a\cos bx + b\sin bx)$$
$$- \frac{e^{ax}}{2(a^2+b^2)^2}\{(a^2-b^2)\cos bx + 2ab\sin bx\}.$$

16. $$\int x^2(\log x)^2 dx = \frac{x^3}{3}\left((\log x)^2 - \frac{2}{3}\log x + \frac{2}{9}\right).$$

17. $$\int \frac{x^3\,dx}{(a+2bx+cx^2)^{\frac{3}{2}}} = \frac{a(2ac-3b^2) + (5ac-6b^2)bx + c(ac-b^2)x^2}{c^2(ac-b^2)\sqrt{(a+2bx+cx^2)}}$$
$$- \frac{3b}{c^2}\int\frac{dx}{\sqrt{(a+2bx+cx^2)}}.$$

18. $$\int \frac{dx}{x^3\sqrt{(a+2bx+cx^2)}} = \left(\frac{3b}{2a^2x} - \frac{1}{2ax^2}\right)\sqrt{(a+2bx+cx^2)}$$
$$+ \frac{(3b^2-ac)}{2a^2}\int\frac{dx}{\sqrt{(a+2bx+cx^2)}}.$$

19. Show that

$$\int x^{m-1}X^n\,dx = \frac{x^m X^n}{m+4n} + \frac{4an}{m+4n}\int x^{m-1}X^{n-1}\,dx + \frac{2nb}{m+4n}\int x^{m+1}X^{n-1}\,dx,$$

where
$$X = a + bx^2 + cx^4.$$

20. $$\int \sin 4x\cos^2 x\,dx = -\frac{1}{6}\cos x(\sin x\sin 4x + 2\cos x\cos 4x) + \frac{1}{24}\cos 4x.$$

21. $$\int \frac{(ab)^5\,d\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^3} = \frac{1}{8}(3a^4 + 3b^4 + 2a^2b^2)\,\phi - \frac{1}{4}(a^4 - b^4)\sin 2\phi + \frac{1}{32}(a^2 - b^2)^2\sin 4\phi,$$

where
$$\tan\phi = \frac{b}{a}\tan\theta.$$

22. $$\int x^2 e^{ax}\cos bx\,dx = \frac{e^{ax}}{c^3}\{c^2x^2\cos(bx - \lambda) - 2cx\cos(bx - 2\lambda) + 2\cos(bx - 3\lambda)\},$$

where
$$a = c\cos\lambda,\quad b = c\sin\lambda.$$

CHAPTER V.

ELLIPTIC INTEGRALS.

77. In this chapter we propose to give a slight account of those integrals which have been called elliptic. This subject is very extensive, and would in fact require a whole treatise for itself. Here, however, we merely investigate some of those properties which are of more immediate use in the geometrical applications of the Integral Calculus.

78. As has been remarked already in Art. 24, the elliptic integrals are the irreducible expressions on which we can make all the integrals of the form $\int \phi(x, y)\,dx$ depend, where y is the square root of a rational integral expression in x containing powers of the variable as far as the fourth degree, namely,

$$y = \sqrt{(a + bx + cx^2 + dx^3 + ex^4)} = \sqrt{X}, \text{ say.}$$

For the purpose of proving this, we commence by transforming X to a simpler form. Now it is shown in treatises on algebra, that X can always be made proportional to $\alpha + \beta z^2 + \gamma z^4$ by a real homographic transformation; that is, by the substitution of $(m + nz)/(p + qz)$ for x. We have, then, $X = (\alpha + \beta z^2 + \gamma z^4)/(p + qz)^4 = Z/(p + qz)^4$, say, and $dx = (np - mq)\,dz/(p + qz)^2$. Now, in the same way as in Art. 38, we see that $\int \phi(x, y)\,dx$ can be written

$$\int \left(P + \frac{Q}{\sqrt{X}}\right) dx,$$

where P and Q are rational functions of x. The first part of this integral is already known; and the second part, namely,

$$\int \frac{Q\,dx}{\sqrt{X}},$$

by the homographic transformation just made use of, becomes

$$\int \frac{R\,dz}{\sqrt{Z}},$$

where R is a rational function of z. But R can be always expressed in the form $\psi(z^2) + z\psi_1(z^2)$; so that we have

$$\int \frac{R\,dz}{\sqrt{Z}} = \int \frac{\{\psi(z^2) + z\psi_1(z^2)\}\,dz}{\sqrt{Z}}$$

$$= \int \frac{\psi(z^2)\,dz}{\sqrt{Z}} + \frac{1}{2}\int \frac{\psi_1(z^2)\,d(z^2)}{\sqrt{Z}};$$

the latter of which expressions becomes, by putting $z^2 = u$,

$$\frac{1}{2}\int \frac{\psi_1(u)\,du}{\sqrt{(\alpha+\beta u+\gamma u^2)}},$$

and is integrable by the methods of Art. 38.

79. We have thus to consider

$$\int \frac{\psi(z^2)\,dz}{\sqrt{Z}},$$

which, by the method of Chapter II., can be made to depend upon integrals of the form

$$\int \frac{z^{2n}\,dz}{\sqrt{Z}} \quad (1),$$

$$\int \frac{dz}{(z^2-a_1)\sqrt{Z}} \quad (2); \qquad \int \frac{dz}{(z^2-a_1)^r\sqrt{Z}} \quad (3);$$

$$\int \frac{(lz^2+m)\,dz}{\{(z^2-a')^2+\beta'^2\}\sqrt{Z}} \quad (4); \qquad \int \frac{(l'z^2+m')\,dz}{\{(z^2-a')^2+\beta'^2\}^r\sqrt{Z}}. \quad (5).$$

But putting Z for X, $2n-3$ for r, and taking $p = \frac{1}{2}$ in Art. 57, we find, by integration,

$$z^{2n-3}\sqrt{Z} = (2n-3)\,a\int \frac{z^{2n-4}\,dz}{\sqrt{Z}} + (2n-2)\,\beta\int \frac{z^{2n-2}\,dz}{\sqrt{Z}}$$

$$+ (2n-1)\,\gamma\int \frac{z^{2n}\,dz}{\sqrt{Z}},$$

from which we get

$$\int \frac{z^{2n}\,dz}{\sqrt{Z}} = \frac{z^{2n-3}\sqrt{Z}}{(2n-1)\,\gamma} - \frac{(2n-2)}{(2n-1)}\frac{\beta}{\gamma}\int \frac{z^{2n-2}\,dz}{\sqrt{Z}}$$

$$- \frac{(2n-3)}{(2n-1)}\frac{a}{\gamma}\int \frac{z^{2n-4}\,dz}{\sqrt{Z}} \quad (6).$$

By this formula of reduction the integral (1) is ultimately made to depend upon

$$\int \frac{z^2\,dz}{\sqrt{Z}} \quad (7) \qquad \text{and} \qquad \int \frac{dz}{\sqrt{Z}} \quad (8).$$

Again, differentiating

$$z\sqrt{Z}\,/\,(z^2-a_1)^{r-1},$$

we get

$$\frac{d}{dz}\left\{\frac{z\sqrt{Z}}{(z^2-a_1)^{r-1}}\right\} = \frac{(z^2-a_1)(a+2\beta z^2+3\gamma z^4)-2(r-1)z^2(a+\beta z^2+\gamma z^4)}{(z^2-a_1)^r\sqrt{Z}}.$$

But the numerator of the fraction on the right-hand side of this equation can evidently be written in the form

$$a_0 + a_1(z^2-a_1) + a_2(z^2-a_1)^2 + a_3(z^2-a_1)^3;$$

so that by integration we may then write down the relation

$$a_0\int\frac{dz}{(z^2-a_1)^r\sqrt{Z}}+a_1\int\frac{dz}{(z^2-a_1)^{r-1}\sqrt{Z}}+a_2\int\frac{dz}{(z^2-a_1)^{r-2}\sqrt{Z}}$$
$$+a_3\int\frac{dz}{(z^2-a_1)^{r-3}\sqrt{Z}}=\frac{z\sqrt{Z}}{(z^2-a_1)^{r-1}}.$$

By this formula of reduction (3) is ultimately made to depend upon the integrals

$$\int\frac{dz}{(z^2-a_1)\sqrt{Z}},\qquad\int\frac{dz}{\sqrt{Z}},\qquad\int\frac{(z^2-a_1)\,dz}{\sqrt{Z}};$$

but the two last are (8) and (7), which occur in the reduction of (1).

In the same way, by means of formulae of reduction, it can be shown that (5), namely,

$$\int\frac{(l'z^2+m')\,dz}{\{(z^2-\alpha')^2+\beta'^2\}^r\sqrt{Z}}$$

depends upon the forms (7) and (8), besides

$$\int\frac{(lz^2+m)\,dz}{\{(z^2-\alpha')^2+\beta'^2\}\sqrt{Z}}.$$

At this stage of our knowledge we cannot demonstrate the possibility of the reduction of the latter integral to the fundamental forms (2), (7), and (8). It suffices, however, to remark here that the integral (4) can be made to depend upon (8), and two integrals of the form (2), in which the quantities corresponding to a_1 are different. The proof of this has to be deferred, as it involves a knowledge of the more advanced parts of the subject.

80. We thus arrive at the result that any integral of the form

$$\int \phi(x, y)\,dx, \text{ where } y^2 = a + bx + cx^2 + dx^3 + ex^4,$$

can be made to depend upon the elementary forms and three irreducible integrals, such as (2), (7), and (8). These three integrals we propose to transform to the simplest forms which they are capable of assuming, namely,

$$F_k(\theta) = \int \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \quad (9), \quad E_k(\theta) = \int \sqrt{(1-k^2\sin^2\theta)}\,d\theta \quad (10),$$

$$\Pi_k(n, \theta) = \int \frac{d\theta}{(1+n\sin^2\theta)\sqrt{(1-k^2\sin^2\theta)}} \quad (11).$$

These three are usually called elliptic integrals of the first, second, and third kinds, respectively. Each of them is supposed to be taken so as to vanish with θ, which is called the amplitude of the integral. The quantity k, which is supposed to be always < 1, is called the modulus; and $\sqrt{(1-k^2)}$, which is denoted by k', is called the complement of the modulus. The constant n which occurs in the third integral is called its parameter, and may have any real value whatever, positive or negative.

In the notation made use of above the suffixes are omitted, when integrals having the same modulus are under consideration, as is usually the case. When $\theta = \frac{1}{2}\pi$, the integrals are called complete, and the values of (9) and (10) are then denoted by K and E, respectively; that is,

$$F_k(\tfrac{1}{2}\pi) = K, \quad E_k(\tfrac{1}{2}\pi) = E;$$

and the corresponding quantities for the integrals with the modulus k' are denoted by K', E'. The quantity $\sqrt{(1-k^2\sin^2\theta)}$

is usually denoted by $\Delta_k(\theta)$, or $\Delta(\theta)$, in investigations in which only one modulus enters.

Tables of the numerical values of $F_k(\theta)$ and $E_k(\theta)$ were calculated by Legendre, who gave the integrals we are considering the name elliptic, from the fact that $E_k(\theta)$ is exactly represented by a portion of the arc of an ellipse.

81. We now consider the actual reduction of (2), (7), and (8) to the forms (9), (10) and (11).

It is easy to see that $\alpha + \beta z^2 + \gamma z^4 = Z$ must be capable of being written in one or other of the forms—

$$(A)\ m^2(z^2 - a^2)(z^2 - b^2), \qquad (B)\ m^2(a^2 - z^2)(b^2 - z^2),$$

$$(C)\ m^2(a^2 - z^2)(z^2 - b^2), \qquad (D)\ m^2(z^2 - a^2)(z^2 + b^2),$$

$$(E)\ m^2(a^2 - z^2)(z^2 + b^2), \qquad (F)\ m^2(z^2 + a^2)(z^2 + b^2),$$

$$(G)\ m^2(z^4 + 2c^2 z^2 \cos 2a + c^4).$$

In case (A) we have

$$\int \frac{dz}{\sqrt{Z}} = \frac{1}{m}\int \frac{dz}{\sqrt{\{(z^2 - a^2)(z^2 - b^2)\}}} = -\frac{1}{ma}\int \frac{d\theta}{\sqrt{(1 - k^2 \sin^2\theta)}};$$

where $\quad z = a/\sin\theta,\ k = b/a,\ b$ being supposed $< a$.

In case (B)—

$$\int \frac{dz}{\sqrt{Z}} = \frac{1}{ma}\int \frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}};$$

where $\quad z = b\sin\theta,\ k = b/a.$

In case (C)—

$$\int \frac{dz}{\sqrt{Z}} = -\frac{1}{m}\int \frac{d\theta}{\sqrt{(a^2\cos^2\theta - b^2)}},$$

where $\quad z = a\cos\theta.$

We have, then,

$$\int \frac{d\theta}{\sqrt{(a^2\cos^2\theta - b^2)}} = \int \frac{d\theta}{\sqrt{(a^2 - b^2 - a^2\sin^2\theta)}} = \frac{1}{a}\int \frac{d\phi}{\sqrt{(1 - k^2\sin^2\phi)}},$$

where

$$a \sin\theta = \sqrt{(a^2 - b^2)} \sin\phi, \quad \text{and} \quad k = \sqrt{(a^2 - b^2)}/a.$$

In case (D)—

$$\int \frac{dz}{\sqrt{Z}} = \frac{1}{m\sqrt{(a^2 + b^2)}} \int \frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}},$$

where $\qquad z = a \sec\theta, \quad k = a/\sqrt{(a^2 + b^2)}.$

In case (E)—

$$\int \frac{dz}{\sqrt{Z}} = -\frac{1}{m\sqrt{(a^2 + b^2)}} \int \frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}},$$

where $\qquad z = a \cos\theta, \quad k = a/\sqrt{(a^2 + b^2)}.$

In case (F)—

$$\int \frac{dz}{\sqrt{Z}} = \frac{1}{ma} \int \frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}},$$

where $\qquad z = b \tan\theta, \quad k = \sqrt{(a^2 - b^2)}/a,$

b being supposed $< a$.

In case (G)—

$$\int \frac{dz}{\sqrt{Z}} = \frac{1}{2mc} \int \frac{d\theta}{\sqrt{(1 - \sin^2\alpha \sin^2\theta)}},$$

where $\qquad z = c \tan \tfrac{1}{2}\theta.$

This case is really included in (F), for it is shown in treatises on Algebra that the form (F) is convertible into (G) by a real homographic transformation.

82. For the integral (7) we have, in case (A),

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ proportional to } \int \frac{d\theta}{\sin^2\theta\,\Delta(\theta)}.$$

Now, differentiating $\cot\theta\Delta(\theta)$, we have

$$\frac{d}{d\theta}\{\cot\theta\Delta(\theta)\} = -\frac{\Delta(\theta)}{\sin^2\theta} - \frac{k^2\cos^2\theta}{\Delta(\theta)}$$

$$= -\frac{1}{\sin^2\Delta(\theta)} + \frac{1}{\Delta(\theta)} - \Delta(\theta).$$

Hence, by integration, we get

$$\int \frac{d\theta}{\sin^2\theta\,\Delta(\theta)} = \int \frac{d\theta}{\Delta(\theta)} - \int \Delta(\theta)\,d\theta - \cot\theta\,\Delta(\theta)$$

$$= F(\theta) - E(\theta) - \cot\theta\Delta(\theta).$$

In case (B), (7) is proportional to

$$\int \frac{\sin^2\theta\,d\theta}{\Delta(\theta)} = \frac{1}{k^2}\{F(\theta) - E(\theta)\}.$$

In case (C)—

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ varies as } \int \sqrt{(1-k^2\sin^2\phi)}\,d\phi = E(\phi).$$

In case (D)—

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ varies as } \int \frac{d\theta}{\cos^2\theta\,\Delta(\theta)}.$$

But we have

$$\frac{d}{d\theta}\{\tan\theta\Delta(\theta)\} = \frac{\Delta(\theta)}{\cos^2\theta} - \frac{k^2\sin^2\theta}{\Delta(\theta)} = \frac{1-k^2}{\cos^2\theta\,\Delta(\theta)}$$

$$- \frac{(1-k^2)}{\Delta(\theta)} + \Delta(\theta);$$

therefore, by integration,

$$\int \frac{d\theta}{\cos^2\theta\,\Delta(\theta)} = \int \frac{d\theta}{\Delta(\theta)} - \frac{1}{k'^2}\int \Delta(\theta)\,d\theta + \frac{1}{k'^2}\tan\theta\,\Delta(\theta)$$

$$= F(\theta) - \frac{1}{k'^2}\{E(\theta) - \tan\theta\,\Delta(\theta)\}.$$

In case (E)—

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ varies as } \int \frac{\cos^2\theta\,d\theta}{\Delta(\theta)} = \frac{1}{k^2}E(\theta) - \frac{k'^2}{k^2}F(\theta).$$

In case (F)—

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ varies as } \int \frac{\tan^2\theta\,d\theta}{\Delta(\theta)} = \int \left(\frac{1}{\cos^2\theta} - 1\right)\frac{d\theta}{\Delta(\theta)},$$

and therefore is reducible to case (D).

In case (G)—

$$\int \frac{z^2\,dz}{\sqrt{Z}} \text{ varies as } \int \frac{\tan^2\frac{1}{2}\theta\,d\theta}{\Delta(\theta)} = \int \frac{(1-\cos\theta)^2}{\sin^2\theta}\,\frac{d\theta}{\Delta(\theta)}$$

$$= \int \left(\frac{2}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} - 1\right)\frac{d\theta}{\Delta(\theta)} = 2\int \frac{d\theta}{\sin^2\theta\,\Delta(\theta)} + \frac{2\Delta(\theta)}{\sin\theta} - F(\theta),$$

and thus is made to depend upon case (A).

83. In all the transformations made use of in the different cases, it will be observed that the integral is always transformed by a substitution of the form

$$z^2 = \frac{m + n\sin^2\theta}{p + q\sin^2\theta};$$

so that in each case

$$\int \frac{dz}{(z^2 - a_1)\sqrt{Z}},$$

by this substitution may be written

$$\int \frac{p + q \sin^2\theta}{m' + n' \sin^2\theta} \frac{d\theta}{\Delta(\theta)},$$

which evidently can at once be made to depend upon (9) and (11), namely, $F(\theta)$ and $\Pi(n, \theta)$.

84. We enter here on a more particular consideration of the integral

$$\int \frac{dx}{\sqrt{X}},$$

and effect the reduction to the form (9) in a different manner. Suppose, first, that X has two real linear factors $x - a$, $x - \beta$, say, so that we may write

$$X = (x - a)(x - \beta) Q, \quad \text{or} \quad (a - x)(x - \beta) Q,$$

where $$Q = \lambda + \mu x + \nu x^2;$$

then putting $$x = \frac{a - \beta z^2}{1 - z^2},$$

we have, from equation (10) in Art. 38,

$$\frac{dx}{\sqrt{\{(x - a)(x - \beta)\}}} = \frac{2dz}{1 - z^2};$$

also we get, evidently,

$$Q = \frac{\lambda' + \mu' z^2 + \nu' z^4}{(1 - z^2)^2},$$

so that we have

$$\frac{dx}{\sqrt{X}} = \frac{dx}{\sqrt{\{(x - a)(x - \beta) Q\}}} = \frac{2dz}{\sqrt{(\lambda' + \mu' z^2 + \nu' z^4)}},$$

which effects the reduction to the form (8).

In the same way, if $X = (\alpha - x)(x - \beta) Q$, we take

$$x = \frac{\alpha + \beta z^2}{1 + z^2}.$$

We have then

$$\frac{dx}{\sqrt{\{(\alpha - x)(x - \beta)\}}} = -\frac{2dz}{1 + z^2},$$

and
$$Q = \frac{\lambda' - \mu' z^2 + \nu' z^4}{(1 + z^2)^2},$$

so that we get

$$\frac{dx}{\sqrt{X}} = \frac{dx}{\sqrt{\{(\alpha - x)(x - \beta) Q\}}} = -\frac{2dz}{\sqrt{(\lambda' - \mu' z^2 + \nu' z^4)}}.$$

It may be observed that in both of these cases the variable x is connected with the amplitude θ by an equation of the form

$$x = (m + n \sin^2\theta) / (p + q \sin^2\theta).$$

Secondly, if we suppose that X has a quadratic factor with imaginary roots, we may put

$$X = \{(x - \alpha)^2 + \beta^2\} Q.$$

We have then

$$\int \frac{dx}{\sqrt{X}} = \int \frac{dy}{\sqrt{\{(y^2 + \beta^2)(l + my + ny^2)\}}},$$

if we put $x = \alpha + y$. Taking now

$$y = \beta \tan(\theta + \gamma),$$

we get
$$\int \frac{dy}{\sqrt{\{(y^2 + \beta^2)(l + my + ny^2)}}$$

$$= \int \frac{d\theta}{\sqrt{\{l \cos^2(\theta + \gamma) + m\beta \sin(\theta + \gamma)\cos(\theta + \gamma) + n\beta^2 \sin^2(\theta + \gamma)\}}}.$$

But the expression under the radical in the latter integral can be written

$$\tfrac{1}{2}(l+n\beta^2)+\tfrac{1}{2}(l-n\beta^2)\cos(2\theta+2\gamma)+\tfrac{1}{2}m\beta\sin(2\theta+2\gamma),$$

or
$$\tfrac{1}{2}(l+n\beta^2)+\tfrac{1}{2}\{(l-n\beta^2)\cos 2\gamma+m\beta\sin 2\gamma\}\cos 2\theta$$
$$+\tfrac{1}{2}\{m\beta\cos 2\gamma-(l-n\beta^2)\sin 2\gamma\}\sin 2\theta,$$

which takes the form $A+B\cos 2\theta$, if we determine γ so that the coefficient of $\sin 2\theta$ vanishes, that is, if we take

$$\tan 2\gamma=\frac{m\beta}{l-n\beta^2}.$$

We get thus

$$\int\frac{dx}{\sqrt{X}}=\int\frac{d\theta}{\sqrt{(A+B\cos 2\theta)}}=\frac{1}{\sqrt{(A+B)}}\int\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}},$$

where $\qquad k^2=2B/(A+B),$ if $A>B,$

and
$$=\frac{1}{\sqrt{(2B)}}\int\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}},$$

where $\qquad k^2=(A+B)/2B,\ \sin\theta=k\sin\phi,$ if $A<B.$

By the methods of this Article, the integral

$$\int\frac{dx}{\sqrt{X}}$$

is reduced in a very simple way to the form $F_k(\theta)$, without making any use of the algebraic theory of the binary quartic.

85. The following investigation may perhaps not prove

unworthy of the attention of the student. Let us consider the reduction of the general integral

$$\int \frac{\phi(x, y)(x\,dy - y\,dx)}{\sqrt{U}}, \tag{12}$$

where $\phi(x, y)$ is the ratio of two homogeneous binary quantics in x, y of the same degree, and

$$U = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.$$

We observe that U is always capable of being expressed as the product of two real quadratic factors, which we shall call u, v. Now we have seen already, in Art. 21, that

$$udv - vdu = \tfrac{1}{2} J(x\,dy - y\,dx),$$

where J is the Jacobian of u, v. We have thus

$$\frac{x\,dy - y\,dx}{\sqrt{U}} = \frac{2(u\,dv - v\,du)}{J\sqrt{(uv)}}.$$

Again, it is easily shown that $\phi(x, y)$ can be expressed in the form

$$\phi_1(u, v) + J\phi_2(u, v),$$

where ϕ_1 is of the degree 0 and ϕ_2 of the degree -2; for if we write down the values of the three quadratics u, v, J, in terms of x, y, we can solve for x^2 and xy, say; and, taking the ratio of these values, we get x/y. Now we have noticed already, in Art. 39, that J is connected with u, v by an identical relation of the form

$$J^2 = \alpha u^2 + \beta uv + \gamma v^2,$$

by means of which we can express x/y, and then $\phi(x, y)$ in the form given above.

The integral (12) is thus transformed into

$$\int 2\{\phi_1(u, v) + J\phi_2(u, v)\}\frac{(u\,dv - v\,du)}{J\sqrt{(uv)}}$$

$$= 2\int\frac{\phi_1(u, v)(u\,dv - v\,du)}{J\sqrt{(uv)}} + 2\int\frac{\phi_2(u, v)(u\,dv - v\,du)}{\sqrt{(uv)}}.$$

Now, if we put $v/u = z^2$, the latter integral becomes

$$4\int\phi_2(1, z^2)\,dz,$$

and is thus reducible to the integration of a rational function of the variable. By the same substitution, the first integral takes the form

$$4\int\frac{\phi_1(1, z^2)\,dz}{\sqrt{(\alpha + \beta z^2 + \gamma z^4)}},$$

after we have put for J its value in terms of u, v. The reduction of the latter expression to the fundamental forms of elliptic integrals is effected by the method used in Arts. 79 and 80.

Examples.

1. $$\int\frac{dx}{\sqrt{(1 - x^3)}} = -\frac{1}{\sqrt[4]{3}}\int\frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}},$$

where $x = 1 - \sqrt{3}\tan^2\frac{1}{2}\theta$, $k = (1 + \sqrt{3})/2\sqrt{2}$.

2. $$\int\frac{dx}{\sqrt{(x^3 - 1)}} = \frac{1}{\sqrt[4]{3}}\int\frac{d\theta}{\sqrt{(1 - k^2\sin^2\theta)}},$$

where $x = 1 + \sqrt{3}\tan^2\frac{1}{2}\theta$, $k = (\sqrt{3} - 1)/2\sqrt{2}$.

3. $$\int\frac{dx}{\sqrt{(1 + x^4)}} = \frac{1}{2}\int\frac{d\theta}{\sqrt{(1 - \frac{1}{2}\sin^2\theta)}},$$

where $x = \tan\frac{1}{2}\theta$.

4. $$\int \frac{dx}{\sqrt{(1-x^4)}} = -\frac{1}{\sqrt{2}} \int \frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}},$$

where $$x = \cos\theta.$$

5. $$\int \sqrt{(a+bx^2+cx^4)}\,dx = \frac{1}{3}\,x\sqrt{(a+bx^2+cx^4)} + \frac{b}{3}\int \frac{x^2\,dx}{\sqrt{(a+bx^2+cx^4)}} + \frac{2a}{3}\int \frac{dx}{\sqrt{(a+bx^2+cx^4)}}.$$

6. $$3k^2 \int \frac{\sin^4\theta\,d\theta}{\sqrt{(1-k^2\sin^2\theta)}} = 2(1+k^2)\int \frac{\sin^2\theta\,d\theta}{\sqrt{(1-k^2\sin^2\theta)}} - \int \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} + \sin\theta\cos\theta\,\Delta(\theta).$$

7. $$\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{k'^2}\,E(\theta) - \frac{k^2\sin\theta\cos\theta}{k'^2\Delta(\theta)}.$$

8. To find the value of

$$\int \frac{d\theta}{(1+n\sin^2\theta)^2\,\Delta(\theta)}$$

we differentiate $\sin\theta\cos\theta\,\Delta(\theta)/(1+n\sin^2\theta)$. We thus obtain

$$2(n+1)(n+k^2)\int \frac{d\theta}{(1+n\sin^2\theta)^2\,\Delta(\theta)} = \frac{n^2\sin\theta\cos\theta\,\Delta(\theta)}{1+n\sin^2\theta} + \{n^2+2n(1+k^2)+3k^2\}\,\Pi(n,\theta) + nE(\theta) - (n+k^2)\,F(\theta).$$

9. $$\int \frac{d\theta}{\sqrt{\sin\theta}} = -\sqrt{2}\int \frac{d\phi}{\sqrt{(1-\frac{1}{2}\sin^2\phi)}},$$

where $$\sin\theta = \cos^2\phi.$$

86. We now proceed to consider the fundamental property of the elliptic integrals of the first kind, namely, that of their addition. This may be stated in the simplest manner as follows. If a, β, γ are three quantities, such that

$$F(a) + F(\beta) = F(\gamma), \qquad (13)$$

then they are also connected by the algebraic relation

$$\cos a \cos \beta - \sin a \sin \beta \,\Delta(\gamma) = \cos \gamma. \tag{14}$$

To prove this, we consider a, β as functions of another variable t, and γ as a constant. Differentiating, then (13), we have

$$\frac{1}{\Delta(a)} \frac{da}{dt} + \frac{1}{\Delta(\beta)} \frac{d\beta}{dt} = 0. \tag{15}$$

As we can assume a to be any function of t we please, we take t, so that

$$\frac{da}{dt} = \Delta(a);$$

we have, then, from (15),

$$\frac{d\beta}{dt} = - \Delta(\beta).$$

Squaring both sides of these equations, and differentiating, we obtain

$$\frac{d^2 a}{dt^2} = - \frac{k^2}{2} \sin 2a, \qquad \frac{d^2 \beta}{dt^2} = - \frac{k^2}{2} \sin 2\beta.$$

By the addition and subtraction of these equations, we get

$$\frac{d^2 p}{dt^2} = - k^2 \sin p \cos q, \qquad \frac{d^2 q}{dt^2} = - k^2 \sin q \cos p,$$

where we have put

$$a + \beta = p, \quad a - \beta = q.$$

We have also

$$\frac{dp}{dt} \frac{dq}{dt} = \left(\frac{da}{dt}\right)^2 - \left(\frac{d\beta}{dt}\right)^2 = - k^2 \sin p \sin q;$$

hence $$\sin q \frac{d^2p}{dt^2} - \frac{dp}{dt}\frac{d(\sin q)}{dt} = 0;$$

from which, by integration, we get

$$\frac{dp}{dt} = m \sin q, \text{ and similarly, } \frac{dq}{dt} = n \sin p, \tag{16}$$

where m and n are constants; therefore,

$$m \sin q \frac{dq}{dt} - n \sin p \frac{dp}{dt} = 0,$$

whence $$m \cos q - n \cos p = c,$$

or $$(m-n) \cos a \cos \beta + (m+n) \sin a \sin \beta = c. \tag{17}$$

But putting $\beta = 0$ in (13), we have $a = \gamma$, so that

$$c = (m-n) \cos \gamma.$$

Now putting for p, q, their values in terms of a, β, in (16), and $\Delta(a)$, $-\Delta(\beta)$, respectively, for da/dt, $d\beta/dt$, we obtain

$$\Delta(a) - \Delta(\beta) = m \sin(a-\beta), \quad \Delta(a) + \Delta(\beta) = n \sin(a+\beta); \tag{18}$$

whence, putting $\beta = 0$, $a = \gamma$, we have

$$m \sin \gamma = \Delta(\gamma) - 1, \quad n \sin \gamma = \Delta(\gamma) + 1.$$

We thus get

$$(m-n) \sin \gamma = -2, \quad (m+n) \sin \gamma = 2\Delta(\gamma), \tag{19}$$

so that (17) becomes, finally,

$$\cos a \cos \beta - \sin a \sin \beta \Delta(\gamma) = \cos \gamma,$$

which is the equation (14) that we proposed to demonstrate. If we change the sign of γ in (14), we get a symmetrical

relation connecting α, β, γ, so that if we clear (14) of radicals, we ought to get a symmetrical result; and this is found to be the case, namely, (14) gives

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=k^2\sin^2\alpha\sin^2\beta\sin^2\gamma.$$

From this we can deduce the two further relations

$$\left.\begin{aligned}\cos\alpha\cos\gamma+\sin\alpha\sin\gamma\,\Delta(\beta)&=\cos\beta,\\ \cos\beta\cos\gamma+\sin\beta\sin\gamma\,\Delta(\alpha)&=\cos\alpha,\end{aligned}\right\}, \tag{20}$$

corresponding to the transcendental equation

$$F(\alpha)+F(\beta)=F(\gamma).$$

87. If we sought an integral of the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0, \tag{21}$$

where
$$X=a+bx+cx^2+dx^3+ex^4,$$
$$Y=a+by+cy^2+dy^3+ey^4,$$

we might proceed in a similar manner. Thus, taking

$$\frac{dx}{dt}=\sqrt{X},\quad\text{we have}\quad\frac{dy}{dt}=-\sqrt{Y},$$

and then
$$\frac{d^2x}{dt^2}=\frac{1}{2}\frac{dX}{dx},\quad\frac{d^2y}{dt^2}=\frac{1}{2}\frac{dY}{dy}.$$

Putting, now, $x=\frac{1}{2}(p+q)$, $y=\frac{1}{2}(p-q)$,

we have
$$\begin{aligned}\frac{d^2p}{dt^2}&=\tfrac{1}{2}\{b+2cx+3dx^2+4ex^3+b+2cy+3dy^2+4ey^3\}\\ &=b+cp+\tfrac{3}{4}d(p^2+q^2)+\tfrac{1}{2}e(p^3+3pq^2);\end{aligned}$$

also, $$\frac{dp}{dt}\frac{dq}{dt} = \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 = X - Y$$

$$= bq + cpq + \tfrac{1}{4} d\,(q^3 + 3p^2 q) + \tfrac{1}{2} e\,(p^3 q + p^3 q).$$

Hence we obtain

$$q\,\frac{d^2 p}{dt^2} - \frac{dp}{dt}\,\frac{dq}{dt} = \tfrac{1}{2} dq^3 + epq^3,$$

an equation which admits of immediate integration by multiplying both sides by $2dp \,/\, q^3 dt$. We thus get

$$\frac{1}{q^2}\left(\frac{dp}{dt}\right)^2 = a + dp + ep^2,$$

where a is an arbitrary constant. Restoring the values of p, q in terms of x, y, and observing that

$$\frac{dp}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \sqrt{X} - \sqrt{Y},$$

we have, finally,

$$\sqrt{X} - \sqrt{Y} = (x - y)\sqrt{\{a + d\,(x + y) + e\,(x + y)^2\}}, \qquad (22)$$

which, as we perceive, constitutes a perfectly general algebraic integral of the differential equation (21).

88. We may notice here the original method by which Euler inferred that there was a general algebraic integral of (21).

Let us consider the equation

$$\phi = A + B\,(x + y)^2 + Cx^2 y^2 + 2H\,(x + y) + 2Gxy$$
$$+ 2Fxy\,(x + y) = 0, \qquad (23)$$

connecting the two variables x, y, that is, the general homo-

geneous relation of the second degree connecting the three quantities 1, $x+y$, xy. Now we may evidently write

$$\phi = uy^2 + 2vy + w = u'x^2 + 2v'x + w', \tag{24}$$

where u, v, w are quadratic expressions in x, and u', v', w' similar functions in y.

Differentiating ϕ then, we get

$$\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy = 0; \tag{25}$$

but from (24),

$$\frac{d\phi}{dx} = 2(u'x + v') = 2\sqrt{(v'^2 - u'w')},$$

and

$$\frac{d\phi}{dy} = 2(uy + v) = 2\sqrt{(v^2 - uw)},$$

the latter values being obtained by solving for x, y, respectively, in (24). Thus (25) becomes

$$\frac{dx}{\sqrt{(v^2 - uw)}} + \frac{dy}{\sqrt{(v'^2 - u'w')}} = 0,$$

which is of the form (21). But comparing $v^2 - uw$, namely,

$$\{Fx^2 + (B + C)x + H\}^2 - (Cx^2 + 2Fx + B)(Bx^2 + 2Hx + A),$$

with X, we have five equations only to determine the six constants A, B, &c., that is, any one of these quantities may be considered as indeterminate. We see thus that ϕ involves an arbitrary constant, and, therefore, that $\phi = 0$ is a general integral of (21).

89. We now proceed to investigate a relation involving the comparison of the elliptic integrals of the second kind.

Given
$$F(a) + F(\beta) = F(\gamma),$$
to show that
$$E(a) + E(\beta) = E(\gamma) + k^2 \sin a \sin\beta \sin\gamma. \qquad (26)$$

Writing $\qquad P = E(a) + E(\beta) - E(\gamma),$

if we consider γ as constant, we have

$$\frac{dP}{dt} = \Delta(a)\frac{da}{dt} + \Delta(\beta)\frac{d\beta}{dt}$$

$$= \frac{(\cos a - \cos\beta\cos\gamma)}{\sin\beta\sin\gamma}\frac{da}{dt} + \frac{(\cos\beta - \cos a\cos\gamma)}{\sin a\sin\gamma}\frac{d\beta}{dt}$$

from (20). Hence

$$2\sin a\sin\beta\sin\gamma\frac{dP}{dt} = -(\cos a - \cos\beta\cos\gamma)\frac{d(\cos a)}{dt}$$
$$-(\cos\beta - \cos a\cos\gamma)\frac{d(\cos\beta)}{dt}$$
$$= \frac{1}{2}\frac{d}{dt}(2\cos a\cos\beta\cos\gamma - \cos^2 a - \cos^2\beta);$$

but we have already shown in Art. 86, that

$$1 - \cos^2 a - \cos^2\beta - \cos^2\gamma + 2\cos a\cos\beta\cos\gamma$$
$$= k^2\sin^2 a\sin^2\beta\sin^2\gamma;$$

therefore $\quad \sin a\sin\beta\sin\gamma\dfrac{dP}{dt} = \dfrac{1}{2}\dfrac{d}{dt}(k^2\sin^2 a\sin^2\beta\sin^2\gamma),$

or
$$\frac{dP}{dt} = \frac{d}{dt}(k^2\sin a\sin\beta\sin\gamma),$$

whence
$$P = k^2\sin a\sin\beta\sin\gamma,$$

no constant being added as P vanishes with a.

We see thus that if we have

$$\cos a \cos \beta - \sin a \sin \beta \Delta(\gamma) = \cos \gamma,$$

then $$E(a) + E(\beta) - E(\gamma) = k^2 \sin a \sin \beta \sin \gamma,$$

or $$G(a) + G(\beta) - G(\gamma) = k^2 \sin a \sin \beta \sin \gamma,$$

where $$G(\theta) = E(\theta) + \lambda F(\theta),$$

λ being any constant quantity.

90. The form of the equation (14) would lead us to infer that the amplitudes a, β, γ can be represented by the sides of a spherical triangle.

Let the sides AB, BC, AC of a spherical triangle ABC be denoted by γ, a, β, respectively; then the opposite angles C, A, B are such that

$$\cos C = \frac{\cos\gamma - \cos a \cos\beta}{\sin a \sin\beta} = -\Delta(\gamma),\ \cos A = \Delta(a),\ \cos B = \Delta(\beta);$$

whence $$k = \frac{\sin C}{\sin\gamma} = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin\beta},$$

which equations agree with the known properties of spherical triangles. We see thus that if a spherical triangle be constructed with one obtuse and two acute angles, such that the ratio of the sine of each angle to the sine of the opposite side is equal to the modulus k; then the three sides a, β, γ are connected by the relation

$$F(a) + F(\beta) = F(\gamma),$$

where γ is the side opposite the obtuse angle.

By means of this representation we can verify the results of the preceding Articles.

Let the sides AC, BC be fixed in position, and AB be of constant length; let $A'B'$ be the consecutive position of AB, and with the point of intersection O as pole, let arcs of small

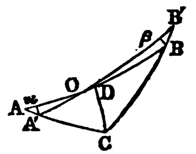

Fig. 1.

circles $A'a$, $B\beta$ be described; then

$$A'O + O\beta = aO + OB,$$

or $$A'\beta = aB\,; \quad \text{also } AB = A'B'\,;$$

therefore $$Aa = B\beta,$$

and in the limit

$$-\cos A d\beta = \cos B da, \quad \text{or} \quad \frac{da}{\cos A} + \frac{d\beta}{\cos B} = 0,$$

which is $$\frac{da}{\Delta(a)} + \frac{d\beta}{\Delta(\beta)} = 0\,;$$

therefore, by integration,

$$F(a) + F(\beta) = F(\gamma),$$

since $\beta = \gamma$, when $a = 0$.

91. We now explain the notation of the inverse elliptic functions. By an inverse elliptic function we mean the expression for x or ϕ in terms of u derived from the equations

$$u = F_k(\phi) = \int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

where $x = \sin\phi$. The notation used is ϕ = the amplitude of $u = amu$, so that

$$x = \sin\phi = \sin amu,\quad \cos\phi = \cos amu,\quad\text{and}\quad \Delta(\phi) = \Delta amu.$$

These three expressions have been respectively simplified into

$$\text{sn}u,\ \text{cn}u,\ \text{dn}u,\quad\text{or}\quad \text{sn}(u, k),\ \text{cn}(u, k),\ \text{dn}(u, k),$$

if the modulus k is specially involved.

The latter forms, which are due to Gudermann, seem to fulfil all the requisites of a good notation, namely, brevity and intelligibility.

Putting $F(\alpha) = u$, $F(\beta) = v$, in (13), we have

$$\gamma = am(u + v),$$

and the formula (14) becomes

$$\text{cn}u\,\text{cn}v - \text{sn}u\,\text{sn}v\,\text{dn}(u + v) = \text{cn}(u + v).$$

Again, from (18) we get

$$(m - n)(\sin^2\alpha - \sin^2\beta) = 2\sin\beta\cos\alpha\Delta(\alpha) - 2\sin\alpha\cos\beta\Delta(\beta);$$

but (19) gives $\quad (m - n)\sin\gamma = -2;$

therefore $\quad \sin\gamma = \dfrac{\sin^2\alpha - \sin^2\beta}{\sin\alpha\cos\beta\Delta(\beta) - \sin\beta\cos\alpha\Delta(\alpha)}.$

Now the product of the two expressions

$$\sin\alpha\cos\beta\Delta(\beta) \pm \sin\beta\cos\alpha\Delta(\alpha),$$

is $\qquad (\sin^2\alpha - \sin^2\beta)(1 - k^2\sin^2\alpha\sin^2\beta),$

so that we have

$$\sin\gamma = \frac{\sin\alpha\cos\beta\,\Delta(\beta) + \sin\beta\cos\alpha\,\Delta(\alpha)}{1-k^2\sin^2\alpha\sin^2\beta},$$

which may be written

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn}u\operatorname{cn}v\operatorname{dn}v + \operatorname{sn}v\operatorname{cn}u\operatorname{dn}u}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}. \tag{27}$$

Changing then the sign of v in the latter equation, and multiplying together the values of $\operatorname{sn}(u+v)$ and $\operatorname{sn}(u-v)$, we obtain

$$\operatorname{sn}(u+v)\operatorname{sn}(u-v) = \frac{\operatorname{sn}^2u - \operatorname{sn}^2v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}. \tag{28}$$

From (18) we get

$$2\sin\alpha\cos\beta\,\Delta(\alpha) - 2\sin\beta\cos\alpha\Delta(\alpha) = (m+n)(\sin^2\alpha - \sin^2\beta)$$

$$= (\sin^2\alpha - \sin^2\beta)\,2\Delta(\gamma)/\sin\gamma, \text{ from (19)};$$

hence we find

$$\Delta(\gamma) = \frac{\sin\alpha\cos\beta\,\Delta(\alpha) - \sin\beta\cos\alpha\,\Delta(\beta)}{\sin\alpha\cos\beta\,\Delta(\beta) - \sin\beta\cos\alpha\,\Delta(\alpha)};$$

and if we multiply both the terms of the latter fraction by

$$\sin\alpha\cos\beta\,\Delta(\beta) + \sin\beta\cos\alpha\,\Delta(\alpha),$$

and then divide by $\sin^2\alpha - \sin^2\beta$, we get

$$\Delta(\gamma) = \frac{\Delta(\alpha)\,\Delta(\beta) - k^2\sin\alpha\sin\beta\cos\alpha\cos\beta}{1-k^2\sin^2\alpha\sin^2\beta},$$

which may be written

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn}u\operatorname{dn}v - k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}u\operatorname{cn}v}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2v}. \tag{29}$$

Substituting this value of $\operatorname{dn}(u+v)$ in the equation

$$\operatorname{cn}(u+v) = \operatorname{cn}u\operatorname{cn}v - \operatorname{sn}u\operatorname{sn}v\operatorname{dn}(u+v),$$

we have

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn}u\operatorname{cn}v - \operatorname{sn}u\operatorname{sn}v\operatorname{dn}u\operatorname{dn}v}{1 - k^2\operatorname{sn}^2u\operatorname{sn}^2v}. \tag{30}$$

From (27) and (30), restoring the a, β, γ notation, we get

$$\tan\gamma = \frac{\tan a\,\Delta(\beta) + \tan\beta\,\Delta(a)}{1 - \tan a\tan\beta\,\Delta(a)\,\Delta(\beta)},$$

which may be written

$$\gamma = \tan^{-1}\{\tan a\,\Delta(\beta)\} + \tan^{-1}\{\tan\beta\,\Delta(a)\}.$$

The latter result, it may be observed, can be readily obtained from fig. 1. Let CD be the arc of a great circle drawn through C perpendicular to AB;

then $$\tan AD = \tan\beta\cos A = \tan\beta\,\Delta(a),$$

and $$\tan BD = \tan a\,\Delta(\beta);$$

hence,

$$\gamma = AD + BD = \tan^{-1}\{\tan\beta\,\Delta(a)\} + \tan^{-1}\{\tan a\,\Delta(\beta)\}.$$

Examples.

1. Show that

$$x^2\sqrt{Y} - y^2\sqrt{X} = (x-y)\sqrt{\{ax^2y^2 + bxy(x+y) + a(x+y)^2\}},$$

where a is an arbitrary constant, is a general integral of the differential equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0.$$

This may be obtained from (22) by putting x^{-1}, y^{-1} for x, y, and reversing the order of the coefficients a, b, &c.

2. Given
$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$
show that
$$\int \frac{x\,dx}{\sqrt{X}} + \int \frac{y\,dy}{\sqrt{Y}} = \int \frac{dp}{\sqrt{(a + dp + ep^2)}},$$
where
$$a \text{ is a constant, and } p = x + y.$$

3. In the same case, show that
$$\int \frac{x^2\,dx}{\sqrt{X}} + \frac{y^2\,dy}{\sqrt{Y}} = \int \frac{p\,dp}{\sqrt{(a + dp + ep^2)}}.$$

4. Also show that
$$\int \frac{dx}{x\sqrt{X}} + \int \frac{dy}{y\sqrt{Y}} = \int \frac{d\varpi}{\sqrt{(a + b\varpi + a\varpi^2)}},$$
where
$$x + y = \varpi xy.$$

5. Given
$$x^2(ay^2 + 2by + c) + 2x(a'y^2 + 2b'y + c') + a''x^2 + 2b''y + c'' = 0;$$
show that
$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$
where
$$X = (ax^2 + 2a'x + a'')(cx^2 + 2c'x + c'') - (bx^2 + 2b'x + b'')^2,$$
$$Y = (ay^2 + 2by + c)(a''y^2 + 2b''y + c'') - (a'y^2 + 2b'y + c')^2.$$

6. Given the base c, and the vertical angle C, of a spherical triangle, show that the sides a, b are connected by the relation
$$\frac{da}{\sqrt{(1 - k^2\sin^2 a)}} + \frac{db}{\sqrt{(1 - k^2\sin^2 b)}} = 0,$$
where
$$k = \sin C / \sin c.$$

7. Show that
$$\left\{\frac{\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u - \operatorname{sn}v\operatorname{cn}v\operatorname{dn}v}{\operatorname{sn}^2 u - \operatorname{sn}^2 v}\right\}^2 =$$
$$\frac{1}{\operatorname{sn}^2(u+v)} + k^2(\operatorname{sn}^2 u + \operatorname{sn}^2 v) - (1 + k^2).$$

This may be obtained from (22), by taking
$$X = x(1 - x)(1 - k^2 x).$$

8. Show that

$$\operatorname{sn}(2u) = \frac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u}{1 - k^2\operatorname{sn}^4 u}.$$

9. Show that

$$\operatorname{dn}(2u) = \frac{1 - 2k^2\operatorname{sn}^2 u + k^2\operatorname{sn}^4 u}{1 - k^2\operatorname{sn}^4 u}.$$

10. Show that

$$\operatorname{cn}(2u) = \frac{1 - 2\operatorname{sn}^2 u + k^2\operatorname{sn}^4 u}{1 - k^2\operatorname{sn}^4 u}.$$

11. If $F(\gamma) = 2F(\alpha)$, then $\tan\frac{1}{2}\gamma = \tan\alpha\Delta(\alpha)$.

12. If $F(\alpha) + F(\beta) = K$; then $\cot\alpha\cot\beta = k'$, and $\Delta(\alpha)\Delta(\beta) = k'$.

13. Show that $$\operatorname{sn}(K - u) = \frac{\operatorname{cn}u}{\operatorname{dn}},$$

and $$\operatorname{cn}(K - u) = \frac{k'\operatorname{sn}u}{\operatorname{dn}u}.$$

14. Show that

$$\operatorname{sn}(u + a) + \operatorname{sn}(u - a) = \frac{2\operatorname{sn}u\operatorname{cn}a\operatorname{dn}a}{1 - k^2\operatorname{sn}^2 a\operatorname{sn}^2 u}.$$

15. Show that $$\operatorname{sn}^2\frac{u}{2} = \frac{1 - \operatorname{cn}u}{1 + \operatorname{dn}u}.$$

16. Show that

$$\frac{1 + \operatorname{cn}(u + v)}{\operatorname{sn}(u + v)} = \frac{\operatorname{sn}v\operatorname{dn}u - \operatorname{sn}u\operatorname{dn}v}{\operatorname{cn}u + \operatorname{cn}v} = \frac{\operatorname{cn}u + \operatorname{cn}v}{\operatorname{sn}u\operatorname{dn}v + \operatorname{sn}v\operatorname{dn}u}.$$

17. Show that

$$\frac{1 - \operatorname{cn}(u + v)}{\operatorname{sn}(u + v)} = \frac{\operatorname{sn}u\operatorname{dn}v + \operatorname{sn}v\operatorname{dn}u}{\operatorname{cn}u + \operatorname{cn}v}.$$

18. Show that

$$\frac{\operatorname{dn}(u + v) + \operatorname{cn}(u + v)}{\operatorname{sn}(u + v)} = \frac{\operatorname{cn}u\operatorname{dn}v + \operatorname{cn}v\operatorname{dn}u}{\operatorname{sn}u + \operatorname{sn}v} = \frac{-k'^2(\operatorname{sn}u - \operatorname{sn}v)}{\operatorname{cn}u\operatorname{dn}v - \operatorname{cn}v\operatorname{dn}u}.$$

19. Show that

$$\frac{\operatorname{cn}(u+v) - \operatorname{dn}(u+v)}{\operatorname{sn}(u+v)} = \frac{\operatorname{cn}u\operatorname{dn}v - \operatorname{cn}v\operatorname{dn}u}{\operatorname{sn}u - \operatorname{sn}v}.$$

20. Show that

$$\operatorname{dn}u\operatorname{dn}v\operatorname{dn}(u + v) = k'^2 + k^2\operatorname{cn}u\operatorname{cn}v\operatorname{cn}(u + v),$$

$$\operatorname{dn}u\operatorname{dn}v = \operatorname{dn}(u + v) + k^2\operatorname{sn}u\operatorname{sn}v\operatorname{cn}(u + v),$$

$$\operatorname{cn}u\operatorname{cn}v\operatorname{dn}(u + v) = \operatorname{dn}u\operatorname{dn}v\operatorname{cn}(u + v) + k'^2\operatorname{sn}u\operatorname{sn}v.$$

92. The elliptic integral of the first kind may be considered as a function more general than the circular and logarithmic integrals, and combining the properties of both; for, corresponding to the extreme values zero and unity of the modulus k, it coincides with each of these functions respectively. Now we know that the inverse circular functions have real periods, and the inverse logarithmic function, namely e^u, imaginary ones. We should be thus led to expect that the inverse elliptic functions have two sets of periods—one real, and the other imaginary; and this is found to be the case, as we now proceed to show. Writing K and $-K$ successively for v in (27), and adding, we get

$$\operatorname{sn}(u+K)+\operatorname{sn}(u-K)=0,$$

as
$$\operatorname{cn}K=\cos\left(\frac{\pi}{2}\right)=0;$$

hence, putting $u+K$ for u,

$$\operatorname{sn}(u+2K)=-\operatorname{sn}u;$$

and, therefore,

$$\operatorname{sn}(u+4K)=-\operatorname{sn}(u+2K)=\operatorname{sn}u,$$

and
$$\operatorname{sn}(u+2mK)=(-1)^m\operatorname{sn}u, \tag{31}$$

where m is any positive or negative integer.

In the same way, from (30) we get

$$\operatorname{cn}(u+K)+\operatorname{cn}(u-K)=0,$$

and
$$\operatorname{cn}(u+2mK)=(-1)^m\operatorname{cn}u. \tag{32}$$

Again, from (29) we have

$$\operatorname{dn}(u+K)-\operatorname{dn}(u-K)=0;$$

whence
$$\operatorname{dn}(u+2mK)=\operatorname{dn}u. \tag{33}$$

We thus perceive the existence of the real periods of the functions sn$u$, cn$u$, dn$u$. To find the imaginary periods, put $\sin\phi = i\tan\theta$ in the equation

$$u = \int \frac{d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}},$$

and we get $u = iv$, where

$$v = \int \frac{d\theta}{\sqrt{(1 - k'^2 \sin^2\theta)}} = F_{k'}(\theta).$$

Now let $\theta = \frac{1}{2}\pi$, then $v = K'$, and

$$\operatorname{sn}(iK') = i \tan \tfrac{1}{2}\pi = \infty.$$

But putting iK' and $-iK'$ successively for v in (27), and subtracting, we get

$$\operatorname{sn}(u + iK') - \operatorname{sn}(u - iK') = \frac{2\operatorname{sn}(iK') \operatorname{cn}u \operatorname{dn}u}{1 - k^2 \operatorname{sn}^2(iK') \operatorname{sn}^2 u},$$

which vanishes when we put $\operatorname{sn}(iK') = \infty$. We thus find

$$\operatorname{sn}(u + 2iK') = \operatorname{sn}u,$$

and
$$\operatorname{sn}(u + 2niK') = \operatorname{sn}u. \tag{34}$$

In the same way, from (30) we get

$$\operatorname{cn}(u + iK') + \operatorname{cn}(u - iK') = 0;$$

whence
$$\operatorname{cn}(u + 2niK') = (-1)^n \operatorname{cn}u. \tag{35}$$

Again, from (29), there is

$$\operatorname{dn}(u + iK') + \operatorname{dn}(u - iK') = 0;$$

whence
$$\operatorname{dn}(u + 2niK') = (-1)^n \operatorname{dn}u. \tag{36}$$

93. From these results we can show that the division of a given function $F(\phi)$ into p equal parts requires, in general,

the solution of an algebraic equation of the degree p^2. For instance, suppose we want to find the value of $x = \operatorname{sn}(u/p)$, being given the value of $\sin\phi = \operatorname{sn} u$; then, because we may change u into $u + 4mK + 2niK'$ without altering the value of $\operatorname{sn} u$, the equation which determines x must equally determine all values included in the equation

$$x = \operatorname{sn}\left\{\frac{u + 4mK + 2niK'}{p}\right\},$$

from which we obviously get different values of x for all the values of m and n from zero to $p - 1$. Hence there are altogether p^2 values of x, of which only p are real.

In the case of division of the complete integral, the degree of the algebraic equation is diminished; for instance, if we wish to find the value of $\operatorname{sn}\left(\frac{K}{3}\right)$, we have $K - u = 2u$, whence

$$\operatorname{sn}(K - u) = \operatorname{sn} 2u;$$

therefore, from Examples 8 and 13, Art. 91,

$$\frac{2\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u} = \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

whence, omitting the factor $\operatorname{cn} u$, we get

$$k^2 x^4 - 2k^2 x^3 + 2x - 1 = 0,$$

putting $\operatorname{sn} u = x$. This equation, it is easy to see, determines the values of the four quantities

$$\operatorname{sn}\left(\frac{K}{3}\right), \quad -\operatorname{sn}\left(K + \frac{2}{3}\, iK'\right), \quad \operatorname{sn}\left(\frac{K \pm 2iK'}{3}\right).$$

94. A most important part of the theory of elliptic integrals is that which treats of the transformation of the integral

of the first kind into another with a different modulus. The general theorem by which this transformation is effected in the most complete manner was first obtained by Jacobi. Two particular cases had, however, been arrived at before, and are generally known as Lagrange's and Legendre's transformations, although the former seems to have been first given by Landen.

The transformation of Landen, which is the only one we propose to notice here, has been, in fact, already given in the different modes adopted for the reduction of the integral (8) to its fundamental form. To show this, let us consider the integral

$$u = \int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}.$$

Taking then $(a - x)(x - \beta) = 1 - x^2$ in the method of Art. 84, we have $a = 1$, $\beta = -1$, and therefore assume

$$x = \frac{1 - z^2}{1 + z^2}.$$

We get then

$$u = -\int \frac{2dz}{\sqrt{\{(1+z^2)^2 - k^2(1-z^2)^2\}}}$$

$$= -2\int \frac{dz}{\sqrt{\{(1+k+(1-k)z^2)\}\{(1-k+(1+k)z^2)\}}},$$

which, by case (F) of Art. 81, becomes

$$-\frac{2}{1+k}\int \frac{d\phi}{\sqrt{(1-\lambda^2\sin^2\phi)}},$$

where $\qquad z = \sqrt{\left(\frac{1-k}{1+k}\right)}\tan\phi, \quad \lambda^2 = \frac{4k}{(1+k)^2}.$

But if we put $\sin\theta$ for x, u is reduced at once to the form $F_k(\theta)$, so that we have thus connected two integrals of the first kind with different moduli. It may be observed, however, that θ and ϕ do not vanish together. In order that the two integrals should both vanish with their amplitudes, we must take

$$x = \operatorname{sn}(K - u) = \cos\theta / \Delta(\theta), \text{ where } \theta = am u.$$

From
$$x = \frac{1 - z^2}{1 + z^2},$$

we have, then,
$$\frac{\cos\theta}{\Delta(\theta)} = \frac{k + \cos 2\phi}{1 + k\cos 2\phi};$$

whence
$$1 - \frac{\cos^2\theta}{\Delta^2(\theta)} = \frac{k'^2\sin^2\theta}{\Delta^2(\theta)} = \frac{k'^2\sin^2 2\phi}{(1 + k\cos 2\phi)^2},$$

or
$$\frac{\sin\theta}{\Delta(\theta)} = \frac{\sin 2\phi}{1 + k\cos 2\phi}. \tag{37}$$

We thus have

$$\tan\theta = \sin 2\phi / (k + \cos 2\phi), \text{ or } \sin(2\phi - \theta) = k\sin\theta; \tag{38}$$

and then, when this relation exists between ϕ and θ, from what we have shown, we have also the transcendental equation

$$F_k(\theta) = \frac{2}{1 + k} F_\lambda(\phi), \tag{39}$$

where
$$\lambda^2 = 4k / (1 + k)^2.$$

95. We may give a simple geometrical illustration of the preceding result. Let ABC be a triangle, of which the base AB is fixed in position, and the side AC is given

in length; then, with the usual notation of plane triangles, we have

$$b \sin C = c \sin B,$$

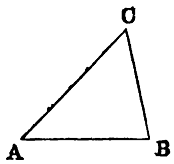

Fig. 2.

and therefore, by differentiation,

$$b \cos C dC = c \cos B dB;$$

but, since $$A + B + C = \pi,$$

there is $$dA + dB + dC = 0;$$

hence $$b \cos C (dA + dB) + c \cos B dB = 0,$$

or $$b \cos C dA + a dB = 0,$$

since $$b \cos C + c \cos B = a.$$

Putting now $\sqrt{(b^2 - c^2 \sin^2 B)}$ for $b \cos C$,

and $\sqrt{(b^2 + c^2 - 2bc \cos A)}$ for a,

we get $$\frac{dB}{\sqrt{(b^2 - c^2 \sin^2 B)}} + \frac{dA}{\sqrt{(b^2 + c^2 - 2bc \cos A)}} = 0;$$

which becomes

$$\frac{d\theta}{\sqrt{(1 - k^2 \sin^2 \theta)}} = \frac{2d\phi}{\sqrt{(1 + k^2 + 2k \cos 2\phi)}},$$

if we put $B = \theta, \quad C = \pi - 2\phi, \quad c = kb.$

Hence, by integration, we get (39); and $c \sin B = b \sin C$ gives the equation (38) connecting θ and ϕ.

96. Let there be two elliptic integrals of the second kind, $E_k(\theta)$ and $E_k(\phi)$, whose moduli and amplitudes are

connected by the same equations as in the two preceding Articles; then, from (37), we have

$$1 + \frac{k^2 \sin^2\theta}{\Delta^2(\theta)} = \frac{1 + k^2 + 2k\cos 2\phi}{(1 + k\cos 2\phi)^2};$$

therefore
$$\frac{1}{\Delta(\theta)} = \frac{\sqrt{(1 + k^2 + 2k\cos 2\phi)}}{1 + k\cos 2\phi};$$

also

$$\frac{\cos\theta}{\Delta(\theta)} = \frac{k + \cos 2\phi}{1 + k\cos 2\phi}.$$

We thus get

$$\Delta(\theta) + k\cos\theta = \frac{1 + k\cos 2\phi + k(k + \cos 2\phi)}{\sqrt{(1 + k^2 + 2k\cos 2\phi)}}$$
$$= \sqrt{(1 + k^2 + 2k\cos 2\phi)};$$

and therefore, from (39),

$$\sqrt{(1 + k^2 + 2k\cos 2\phi)}d\phi = \frac{(1 + k^2 + 2k\cos 2\phi)d\phi}{\sqrt{(1 + k^2 + 2k\cos 2\phi)}}$$
$$= \frac{\{\Delta(\theta) + k\cos\theta\}^2 d\theta}{2\Delta(\theta)}$$
$$= \frac{1}{2}\left\{\Delta(\theta) + 2k\cos\theta + \frac{k^2\cos^2\theta}{\Delta(\theta)}\right\}d\theta$$
$$= \frac{1}{2}\left\{2\Delta(\theta) + 2k\cos\theta - \frac{k'^2}{\Delta(\theta)}\right\}d\theta.$$

Hence, by integration, observing that

$$\sqrt{(1 + k^2 + 2k\cos 2\phi)} = (1 + k)\Delta_\lambda(\phi),$$

we have

$$(1 + k)E_\lambda(\phi) = E_k(\theta) + k\sin\theta - \tfrac{1}{2}k'^2 F_k(\theta). \qquad (40)$$

It may be noticed that by this formula an integral of the first kind can be expressed in terms of two integrals of the second kind with different moduli.

97. In the same way as for the integrals of the second kind, we can find a formula for the comparison of those of the third kind.

Given the equation

$$\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta\Delta(\gamma),$$

we have already shown that

$$F(\alpha) + F(\beta) = F(\gamma)$$

$$E(\alpha) + E(\beta) - E(\gamma) = k^2\sin\alpha\sin\beta\sin\gamma.$$

Writing now

$$\Pi(n, \alpha) + \Pi(n, \beta) - \Pi(n, \gamma) = P,$$

we have

$$\frac{1}{1 + n\sin^2\alpha}\,\frac{1}{\Delta(\alpha)}\,\frac{d\alpha}{dt} + \frac{1}{1 + n\sin^2\beta}\,\frac{1}{\Delta(\beta)}\,\frac{d\beta}{dt} = \frac{dP}{dt},$$

or

$$\frac{1}{1 + n\sin^2\alpha} - \frac{1}{1 + n\sin^2\beta} = \frac{dP}{dt},$$

if we consider γ as constant, and take, as in Art. 86,

$$\frac{d\alpha}{dt} = \Delta(\alpha), \quad \frac{d\beta}{dt} = -\Delta(\beta).$$

Hence,

$$\frac{dP}{dt} = \frac{n(\sin^2\beta - \sin^2\alpha)}{1 + n(\sin^2\alpha + \sin^2\beta) + n^2\sin^2\alpha\sin^2\beta};$$

but differentiating the second of the two equations given above, substituting for $d\alpha/dt$ and $d\beta/dt$, and dividing by k^2, we get

$$\sin^2\beta - \sin^2\alpha = \frac{d}{dt}(\sin\alpha\sin\beta\sin\gamma).$$

We thus have

$$\frac{dP}{dt} = \frac{dz}{dt}\,\frac{n\sin\gamma}{1 + n^2z^2 + n(\sin^2\alpha + \sin^2\beta)},$$

where $\sin a \sin \beta = z$. Now from the equation

$$\cos \gamma = \cos a \cos \beta - z\Delta(\gamma),$$

we have

$$\{\cos \gamma + z\Delta(\gamma)\}^2 = 1 - \sin^2 a - \sin^2 \beta + \sin^2 a \sin^2 \beta ;$$

therefore

$$\sin^2 a + \sin^2 \beta = 1 + z^2 - \{\cos \gamma + z\Delta(\gamma)\}^2$$

$$= \sin^2 \gamma (1 + k^2 z^2) - 2 \cos \gamma \Delta(\gamma) z.$$

Hence $$\frac{dP}{dz} = \frac{n \sin \gamma}{1 + n^2 z^2 + n \sin^2 \gamma (1 + k^2 z^2) - 2n \cos \gamma \Delta(\gamma) z};$$

therefore, by integration,

$$P = \frac{1}{\sqrt{\mu}} \tan^{-1} \left\{ \frac{(n + k^2 \sin^2 \gamma) z - \cos \gamma \Delta(\gamma)}{\sin \gamma \sqrt{\mu}} \right\}$$

$$+ \frac{1}{\sqrt{\mu}} \tan^{-1} \left(\frac{\cot \gamma \Delta(\gamma)}{\sqrt{\mu}} \right),$$

where $\mu = (n + 1)(n + k^2) / n$, and the constant is determined so that P should vanish with z. Restoring then the value of z, and simplifying by means of the equation of condition, we find

$$P = \frac{1}{\sqrt{\mu}} \tan^{-1} \left(\frac{n \sqrt{\mu} \sin a \sin \beta \sin \gamma}{1 + n - n \cos a \cos \beta \cos \gamma} \right);$$

so that we have finally

$$\Pi(n, a) + \Pi(n, \beta) - \Pi(n, \gamma)$$

$$= \frac{1}{\sqrt{\mu}} \tan^{-1} \left(\frac{n \sqrt{\mu} \sin a \sin \beta \sin \gamma}{1 + n - n \cos a \cos \beta \cos \gamma} \right) \quad (41)$$

when a, β, γ are connected by the algebraic relation (14), or the transcendental equation (13).

98. If the quantity μ is negative, the circular expression must be replaced by a logarithm. We see thus that the integrals of the third kind belong to one or other of two species, which are as distinct from each other as logarithms and circular functions.

The parameters in each of these cases are called circular and logarithmic respectively; and in the first case it is usual to put $-1+k'^2\sin^2\lambda$, and in the second case, $-k^2\sin^2\lambda$ for n, the corresponding values of μ being then

$$k'^4\sin^2\lambda\cos^2\lambda/\Delta_{k'}^2(\lambda) \quad \text{and} \quad -\cot^2\lambda\,\Delta_k^2(\lambda).$$

99. We investigate here a method of finding the algebraic relation corresponding to the transcendental equation $\Sigma u = 0$, or some constant quantity, where u is an integral of the first kind, and the summation is taken with regard to an even number of the integrals u_1, u_2, &c. This is, in fact, the particular application of Abel's Theorem to the Elliptic Integrals of the first kind.

Let X be a given expression having its usual meaning in this Chapter, and let

$$\phi = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$

$$\psi = b_0 + b_1x + b_2x^2 + \ldots + b_{n-2}x^{n-2};$$

that is, let ϕ and ψ be polynomials in x of the degrees n and $n-2$, respectively. Now let $x_1, x_2, \ldots x_{2n}$ be the $2n$ roots of the equation

$$\phi^2 - \psi^2X = f(x),\text{ say, } = 0; \qquad (42)$$

so that we may then write

$$f(x) = \phi^2 - \psi^2X = \lambda(x-x_1)(x-x_2)\ldots(x-x_{2n}).$$

Equating then the coefficients in this identity, and elimi-

nating λ, we have $2n$ equations connecting $x_1, x_2, \ldots x_{2n}$ with a_0, a_1, &c., and b_0, b_1, &c.; but of the latter quantities, which are $2n$ in number, only the ratios are involved, so that they may be all eliminated from the equations just mentioned. We see thus that there is a single relation connecting the $2n$ quantities $x_1, x_2, \ldots x_{2n}$; and this, we proceed to show, is the algebraic equivalent of a certain transcendental equation of the form mentioned above. Let us assume all the quantities a_0, a_1, &c., b_0, b_1, &c., and, therefore, also x_1, x_2, &c., to be functions of a variable t; then the equation (42) connects any one of the quantities, x_r, say, with t, and by differentiation gives

$$\frac{df(x_r)}{dx_r}\frac{dx_r}{dt} + 2\phi\frac{d\phi}{dt} - 2\psi X_r\frac{d\psi}{dt} = 0.$$

But from $f(x_r) = 0$, $\phi = \epsilon\psi\sqrt{X_r}$, where $\epsilon = \pm 1$,

and $df(x_r)/dx_r = \lambda(x_r - x_1)(x_r - x_2)(x_r - x_{2n}) = f'(x_r)$, say,

so that we have

$$f'(x_r)\frac{dx_r}{dt} + 2\epsilon\sqrt{X_r}\left(\psi\frac{d\phi}{dt} - \phi\frac{d\psi}{dt}\right) = 0,$$

or
$$\frac{1}{\sqrt{X_r}}\frac{dx_r}{dt} + \frac{2\epsilon}{f'(x_r)}\left(\psi\frac{d\phi}{dt} - \phi\frac{d\psi}{dt}\right) = 0.$$

Giving r then all the values from 1 to $2n$ inclusive, and summing, we get

$$\Sigma_1^{2n}\frac{\epsilon_r}{\sqrt{X_r}}\frac{dx_r}{dt} + 2\Sigma_1^{2n}\frac{\psi\dfrac{d\phi}{dt} - \phi\dfrac{d\psi}{dt}}{f'(x_r)} = 0.$$

Now $\psi \dfrac{d\phi}{dt} - \phi \dfrac{d\psi}{dt}$ is of the form

$$c_0 + c_1 x + c_2 x^2 + \ldots + c_{2n-2} x^{2n-2} = \chi(x), \text{ say,}$$

and we know that

$$\Sigma_1^{2n} \frac{\chi(x_r)}{f'(x_r)} = 0.$$

Hence $$\Sigma_1^{2n} \frac{\epsilon_r}{\sqrt{X_r}} \frac{dx_r}{dt} = 0;$$

therefore integrating, and putting

$$\int \frac{dx_r}{\sqrt{X_r}} = u_r,$$

we obtain

$$\epsilon_1 u_1 + \epsilon_2 u_2 + \ldots + \epsilon_{2n} u_{2n} = \text{a constant,} \tag{43}$$

where ϵ_1, ϵ_2, &c., have either of the values ± 1, according to the terms to which they are annexed.

The corresponding algebraic relation can be readily obtained in the form of a determinant; for, substituting x_1, x_2, $\ldots x_{2n}$ successively in $\phi - \epsilon\psi\sqrt{X} = 0$, we get $2n$ equations which are linear and homogeneous in a_0, a_1, &c., b_0, b_1, &c., so that these quantities can be eliminated at once, the result being thus expressed in a determinant form involving the radicals $\sqrt{X_1}$, &c.

100. As an example, let us take $n = 2$; then giving X the form $x(1-x)(1-k^2x)$, we have

$$f(x) = (a_0 + a_1 x + a_2 x^2)^2 - b_0^2 x(1-x)(1-k^2x)$$

$$= \lambda(x-x_1)(x-x_2)(x-x_3)(x-x_4).$$

The relation connecting the inverse functions may be most simply obtained in this case as follows:—Comparing the coefficients of x^4 and the absolute terms, after extracting square roots, we get

$$a_2 = \sqrt{\lambda}, \quad a_0 = s_1 s_2 s_3 s_4 \sqrt{\lambda},$$

where we have put s_r for $\operatorname{sn} u_r = \sqrt{x_r}$. Again, putting 1 and $1/k^2$ successively for x, we obtain, in the same way,

$$a_0 + a_1 + a_2 = c_1 c_2 c_3 c_4 \sqrt{\lambda}, \quad a_0 k^4 + a_1 k^2 + a_2 = d_1 d_2 d_3 d_4 \sqrt{\lambda},$$

where we have put c_r and d_r for $\operatorname{cn} u_r$ and $\operatorname{dn} u_r$, respectively. Hence, eliminating a_0, a_1, a_2, from these four equations, we have

$$k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4 - k'^2 = 0, \qquad (44)$$

which relation, we have thus proved, holds when

$$u_1 + u_2 + u_3 + u_4 = 0,$$

the constant being zero, because (44) is satisfied by taking

$$u_1 + u_2 = 0, \quad u_3 + u_4 = 0.$$

The formula (44) seems to have been first given by Legendre (*Traité des Fonctions Elliptiques*, t. iii., p. 193).

Examples.

1. Show that

$$K = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \text{\&c.}\right\}.$$

This series is obtained by expanding $\sqrt{(1 - k^2 \sin^2 \theta)}$ by the binomial theorem, integrating each term by Art. 58, and then putting $\theta = \frac{1}{2}\pi$.

2. Show that

$$E = \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{k^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{k^6}{5} - \&c.\right\}.$$

3. $$\int \frac{dx}{\sqrt{\{(\alpha - x)(x - \beta)(x + \gamma)\}}} = -\frac{2}{\sqrt{(\alpha + \gamma)}} \int \frac{d\theta}{\sqrt{(1 - k^2 \sin^2\theta)}},$$

where $x = \alpha \cos^2\theta + \beta \sin^2\theta, \quad (\alpha - \beta)/(\alpha + \gamma) = k^2.$

4. $$\int \frac{dx}{\sqrt{(xQ)}} = \frac{1}{\sqrt[4]{(a^2 + \beta^2)}} \int \frac{d\theta}{\sqrt{(1 - \cos^2\gamma \sin^2\theta)}},$$

where $$Q = (x - a)^2 + \beta^2,$$

$$x = \sqrt{(a^2 + \beta^2)} \tan^2 \tfrac{1}{2}\theta, \qquad \tan\gamma = \beta / a.$$

5. Show that $$Eam\,(2u) = 2Eamu - \frac{k^2 \operatorname{sn}^3 u \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^4 u}.$$

6. Show that $$Eamu + Eam\,(K - u) = E + \frac{k^2 \operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

7. Show that $$\operatorname{dn}\left(\frac{K}{2}\right) = \operatorname{cn}\left(\frac{K}{2}\right), \quad \operatorname{sn}\left(\frac{K}{2}\right) = \sqrt{k'}.$$

8. Show that $$Eam\left(\frac{K}{2}\right) = \tfrac{1}{2}E + \tfrac{1}{2}(1 - k').$$

9. $$Eam\,(u + a) + Eam\,(u - a) = 2Eamu - \frac{2k^2 \operatorname{sn}^2 u \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}{1 - k^2 \operatorname{sn}^2 a \operatorname{sn}^2 u}.$$

10. Show that $$k \operatorname{sn} u \operatorname{sn}(u + iK') = 1,$$

$$\operatorname{sn}\,(u + K + iK') = \operatorname{dn} u \,/\, (k \operatorname{cn} u).$$

11. Given $\sin\theta = i \tan\phi$, show that

$$E_k\,(\theta) = i\,\{\tan\phi \Delta_{k'}(\phi) + F_{k'}(\phi) - E_{k'}(\phi)\}.$$

12. Show that $Eam\,(iK') = \infty, \quad Eam\,(2iK') = 2i\,(K' - E').$

13. $$Eam\,(u + iK') = Eamu + \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} + i\,(K' - E').$$

We have $$Eam\,(u + iK') = \int \{1 - k^2 \operatorname{sn}^2(u + iK')\}\,du$$

$$= \int \left(1 - \frac{1}{\operatorname{sn}^2 u}\right) du, \quad \text{from Ex. 10,}$$

or $$Eamu + (\operatorname{cn} u \operatorname{dn} u) / \operatorname{sn} u + C \text{ from Art. 82.}$$

To determine C we have—

$$Eam(u+2iK') = Eamu + 2C, \text{ since } \frac{\text{cn}u\,\text{dn}u}{\text{sn}u} + \frac{\text{cn}(u+iK')\,\text{dn}(u+iK')}{\text{sn}(u+iK')} = 0;$$

whence $$2C = Eam(2iK') = 2i(K'-E'),$$

from the preceding example.

14. Show that $$Eam(K+iK') = E + i(K'-E').$$

15. Show that $$\text{sn}\left(\frac{iK'}{2}\right) = \frac{i}{\sqrt{k}},$$

$$\text{sn}\left(\frac{K+iK'}{2}\right) = \frac{\sqrt{(1+k)} - i\sqrt{(1-k)}}{\sqrt{(2k)}}.$$

16. Show that $$Eam\left(\frac{iK'}{2}\right) = \tfrac{1}{2}i(1+k'+K'-E').$$

17. Given $$\tan(\theta-\phi) = k'\tan\phi,$$

$$\lambda = (1-k')/(1+k'),$$

show that $$(1+k')E_k(\phi) = E\lambda(\theta).$$

18. To show that

$$\Pi(n,\phi) + \Pi(k^2/n,\phi) - F(\phi) = \frac{1}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{\mu}\tan\phi}{\Delta(\phi)}\right\}:$$

Let $$P = \frac{1}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{\mu}\tan\phi}{\Delta(\phi)}\right\};$$

then $$\frac{dP}{d\phi} = \frac{\dfrac{d}{d\phi}\left\{\dfrac{\tan\phi}{\Delta(\phi)}\right\}}{1+\dfrac{\mu\tan^2\phi}{\Delta^2(\phi)}}$$

$$= \frac{\Delta^2(\phi)\cos^2\phi\{\sec^2\phi/\Delta(\phi) + k^2\sin^2\phi/\Delta^3(\phi)\}}{\cos^2\phi(1-k^2\sin^2\phi) + (1+n)(1+k^2/n)\sin^2\phi}$$

$$= \frac{1-k^2\sin^4\phi}{1+(n+k^2/n)\sin^2\phi + k^2\sin^4\phi}\,\frac{1}{\Delta(\phi)}$$

$$= \left\{\frac{1}{1+n\sin^2\phi} + \frac{1}{1+k^2\sin^2\phi/n} - 1\right\}\frac{1}{\Delta(\phi)}.$$

Hence, by integration, the relation given above follows at once.

19. $$\int \frac{1}{1+k^2\sin^4\phi}\,\frac{d\phi}{\Delta(\phi)} = \tfrac{1}{2}F(\phi) + \frac{1}{2\sqrt{(1+k^2)}}\tan^{-1}\left\{\frac{\tan\phi\sqrt{(1+k^2)}}{\Delta(\phi)}\right\}.$$

20. $$\int \frac{1}{1+k\sin^2\phi}\,\frac{d\phi}{\Delta(\phi)} = \tfrac{1}{2}F(\phi) + \frac{1}{2(1+k)}\tan^{-1}\left\{\frac{(1+k)\tan\phi}{\Delta(\phi)}\right\}.$$

21. Let $\mathrm{sn}^2 u_1$, $\mathrm{sn}^2 u_2$, $\mathrm{sn}^2 u_3$, be the values of the roots of the cubic equation

$$(\alpha+\beta x)^2 - x(1-x)(1-k^2x) = 0;$$

then show that

$$u_1 \pm u_2 \pm u_3 = 2mK + (2n+1)iK'.$$

22. Hence show that, in the same case, the roots of

$$x(\alpha+\beta x)^2 - (1-x)(1-k^2x) = 0$$

are connected by the relation

$$u_1 \pm u_2 \pm u_3 = 2mK + 2niK'.$$

CHAPTER VI.

DEFINITE INTEGRALS.

101. We commence by considering the process of integration regarded as a summation, in connexion with which point of view, as has been already remarked, the name integral has arisen. If we suppose any quantity, y say, to vary continuously by indefinitely small increments, commencing with a value y_2 and increasing to a value y_1, the total change of value of y between these limits is obviously $y_1 - y_2$; but this is equal to the sum of all the small partial increments. This result is represented by the notation

$$\int_{y_2}^{y_1} dy = y_1 - y_2,$$

where y_1, y_2 are called the *superior* and *inferior* limits of integration, respectively.

Putting now $y = f(x)$, then $dy = f'(x)\,dx$; and if a, b are the values of x corresponding to the values y_1, y_2 of y, we have

$$\int_b^a f'(x)\,dx = f(a) - f(b).$$

That is, the *definite* integral of $f'(x)dx$, in which x is taken between the limits a and b, is equal to the difference of the values of the general or *indefinite* integral corresponding to $x = a$, and $x = b$; so that we can always find the value of the definite integral, when that of the indefinite integral is

known. The importance, however, of the theory of definite integrals arises from the fact that a very large number of these integrals, taken between certain limits, can have their values assigned, while the value of the indefinite integral in each case remains unknown.

102. We consider here more particularly the process of integration from the point of view of the preceding Article. Let $f(x)$ denote a function of x, which remains finite and continuous for all values of x between the limits x_n and x_0, and let $x_n - x_0$ be divided into n intervals,

$$x_1 - x_0, \quad x_2 - x_1, \ldots x_n - x_{n-1}.$$

Now, by the Differential Calculus, we have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

in the limit, when $x_1 = x_0$, so that we may put

$$f(x_1) - f(x_0) = (x_1 - x_0)\{f'(x_0) + \varepsilon_0\},$$

where ε_0 becomes indefinitely small along with $x_1 - x_0$. We may thus write down the equations:

$$f(x_1) - f(x_0) = (x_1 - x_0)\{f'(x_0) + \varepsilon_0\},$$

$$f(x_2) - f(x_1) = (x_2 - x_1)\{f'(x_1) + \varepsilon_1\},$$

$$f(x_{r+1}) - f(x_r) = (x_{r+1} - x_r)\{f'(x_r) + \varepsilon_r\},$$

$$f(x_n) - f(x_{n-1}) = (x_n - x_{n-1})\{f(x_{n-1}) + \varepsilon_{n-1}\},$$

where $\varepsilon_0, \varepsilon_1 \ldots \varepsilon_{n-1}$ all vanish along with the intervals to which they are annexed.

Hence, by addition, we obtain

$$f(x_n) - f(x_0) = \Sigma_0^{n-1} (x_{r+1} - x_r) f'(x_r) + \Sigma_0^{n-1} (x_{r+1} - x_r) \epsilon_r.$$

Now if ϵ is the greatest of the quantities ϵ_r, the latter sum is evidently less than $(x_n - x_0)\epsilon$, and, therefore, vanishes ultimately when the number of the intervals is increased indefinitely. We thus have

$$f(x_n) - f(x_0) = \text{the limit of } \Sigma_0^{n-1} (x_{r+1} - x_r) f'(x_r), \quad (1)$$

or, as it is written,

$$\int_{x_0}^{x_n} f'(x)\, dx.$$

103. If the intervals are equidistant, we have

$$f(x_n) - f(x_0) = \text{the limit of } h\,\Sigma_0^n f'(x_0 + rh),$$

when h is indefinitely diminished and n increased, nh being taken equal to $x_n - x_0$. This result may be also stated as follows:—If $nh = a - b$, then

$$\int_b^a f(x)\, dx = \text{the limit of the infinite series}$$

$$h\{f(b) + f(b+h) + f(b+2h) + \ldots + f(b+nh)\}, \quad (2)$$

when n is indefinitely increased.

EXAMPLES.

1. $$\int_0^1 x^m\,dx = \frac{1}{m+1}.$$

2. $$\int_0^a \frac{dx}{a^2+x^2} = \frac{1}{2}\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{4a}.$$

3. $$\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{\pi}{4}.$$

4. $$\int_b^a \frac{dx}{\sqrt{\{(a-x)(x-b)\}}} = \pi.$$

5. $$\int_b^a \sqrt{\{(a-x)(x-b)\}}\,dx = \frac{\pi}{8}(a-b)^2.$$

6. $$\int_0^\pi \frac{d\theta}{a^2+b^2-2ab\cos\theta} = \frac{\pi}{a^2-b^2}.$$

7. $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta} = \frac{\pi}{2\alpha\beta}.$$

8. $$\int_0^{\frac{\pi}{2}} \frac{d\theta}{(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)^2} = \frac{\pi(\alpha^2+\beta^2)}{4\alpha^3\beta^3}.$$

9. $$\int_0^a \frac{dx}{\sqrt{(a^2-x^2)}} = \frac{\pi}{2}.$$

10. $$\int_0^1 \frac{dx}{1+2x\cos\alpha+x^2} = \frac{1}{2}\int_0^\infty \frac{dx}{1+2x\cos\alpha+x^2} = \frac{\alpha}{2\sin\alpha}.$$

11. $$\int_0^\infty \epsilon^{-ax}\sin mx\,dx = \frac{m}{m^2+a^2}.$$

12. $$\int_0^\infty \epsilon^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}.$$

13. $$\int_0^\pi \cos mx\cos nx\,dx = 0,$$

$$\int_0^\pi \sin mx\sin nx\,dx = 0,$$

for all integer values of m and n, except when $m = n$, and then each $= \frac{1}{2}\pi$.

104. We notice here a few properties of definite integrals. We have

$$\int_b^a f'(x)dx = f(a) - f(b),$$

and
$$\int_a^b f'(x)dx = f(b) - f(a) = -\int_b^a f'(x)\,dx\,;$$

that is, the interchange of the limits changes the sign of a definite integral.

If we substitute $a - y$ for x in the definite integral

$\int_0^a f(x)dx$, it becomes $-\int_a^0 f(a-y)dy = \int_0^a f(a-y)dy$.

This result may evidently be written

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)dx. \tag{3}$$

As an example, let $f(x) = \phi(\sin x)$, and $a = \pi/2$;

then we have
$$\int_0^{\frac{\pi}{2}} \phi\,(\sin x)dx = \int_0^{\frac{\pi}{2}} \phi\,(\cos x)\,dx. \tag{4}$$

More generally we obtain

$$\int_b^a f(x)dx = \int_b^a f(a+b-x)dx. \tag{5}$$

Again, it is evident that, instead of taking an integral between the limits a and b of the variable, we may take it between the limits a and c, and between c and b, and add the results together, and we shall obtain the same value for the definite integral.

Hence we can show that

$$\int_0^a f(x)dx = 2\int_0^{\frac{1}{2}a} f(x)\,dx, \tag{6}$$

if $\qquad f(x)$ is such that $f(a-x) = f(x)$;

for we have

$$\int_0^a f(x)dx = \int_{\frac{1}{2}a}^a f(x)dx + \int_0^{\frac{1}{2}a} f(x)dx;$$

but the first of the integrals on the right-hand side becomes

$$-\int_{\frac{1}{2}a}^0 f(a-x)\,dx, \quad \text{or} \quad \int_0^{\frac{1}{2}a} f(a-x)\,dx,$$

by putting $a-x$ for x; hence, by the property of the function $f(x)$, (6) follows.

Examples.

1. If $f(a-x) = -f(x)$, and $f(x)$ remains finite between $x=0$ and $x=a$ shew that

$$\int_0^a f(x)\,dx = 0.$$

2. To evaluate $\qquad u = \int_0^{\frac{1}{2}\pi} \log(\sin\theta)\,d\theta.$

We have $\qquad u = \int_0^{\frac{1}{2}\pi} \log(\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi} \log(\cos\theta)\,d\theta;$

therefore

$$2u = \int_0^{\frac{1}{2}\pi} \log(\tfrac{1}{2}\sin 2\theta)\,d\theta = \tfrac{1}{2}\pi\log(\tfrac{1}{2}) + \int_0^{\frac{1}{2}\pi} \log(\sin 2\theta)\,d\theta;$$

but putting $2\theta = \phi$, the latter integral becomes

$$\tfrac{1}{2}\int_0^{\pi} \log(\sin\phi)\,d\phi,$$

which, by (6), equals $\qquad \int_0^{\frac{1}{2}\pi} \log(\sin\phi)\,d\phi$, or u;

therefore $\qquad u = \int_0^{\frac{1}{2}\pi} \log(\sin\theta)\,d\theta = \frac{\pi}{2}\log\left(\frac{1}{2}\right).$

3. $$\int_0^{\pi} \frac{\theta\sin\theta\,d\theta}{1+c^2\cos^2\theta} = \int_0^{\pi} \frac{(\pi-\theta)\sin\theta\,d\theta}{1+c^2\cos^2\theta};$$

therefore $\qquad \int_0^{\pi} \frac{\theta\sin\theta\,d\theta}{1+c^2\cos^2\theta} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin\theta\,d\theta}{1+c^2\cos^2\theta} = \frac{\pi}{c}\tan^{-1}c.$

4. $$\int \theta \sin\theta \log(\sin\theta)\, d\theta = -\pi(1-\log 2).$$

5. $$\int_0^{\pi} \frac{\theta\, d\theta}{1+c^2-2c\cos\theta} = \frac{\pi^2}{2(1-c^2)}.$$

6. $$(1-e^{-2a\pi})\int_0^{\infty} e^{-ax} f(\sin x, \cos x)\, dx = \int_0^{2\pi} e^{-ax} f(\sin x, \cos x)\, dx.$$

We have $$\int_0^{4\pi} e^{-ax} u dx = \int_{2\pi}^{4\pi} e^{-ax} u dx + \int_0^{2\pi} e^{-ax} u dx$$

$$= \int_0^{2\pi} e^{-a(x+2\pi)} u dx + \int_0^{2\pi} e^{-ax} u dx,$$

where $$f(\sin x, \cos x) = u;$$

that is, $$\int_0^{4\pi} e^{-ax} u dx = (1+e^{-2a\pi})\int_0^{2\pi} e^{-ax} u dx.$$

Proceeding in the same way, we get

$$\int_0^{6\pi} e^{-ax} u dx = (1+e^{-2a\pi}+e^{-4a\pi})\int_0^{2\pi} e^{-ax} u dx,$$

and, ultimately,

$$\int_0^{2n\pi} e^{-ax} u dx = \{1+e^{-2a\pi}+e^{-4a\pi}+\ldots+e^{-2(n-1)a\pi}\}\int_0^{2\pi} e^{-ax} u dx.$$

Hence, making $n=\infty$, and summing the infinite series between the brackets, we obtain the required result.

105. We now consider the important definite integral

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n d\theta, \quad \text{or} \quad \int_0^{\frac{1}{2}\pi} (\cos\theta)^n d\theta,$$

for these integrals are evidently equal by (4).

From (35), Art. 58, we have

$$\int (\sin\theta)^n d\theta = -\frac{\cos\theta (\sin\theta)^{n-1}}{n} + \frac{(n-1)}{n}\int (\sin\theta)^{n-2} d\theta;$$

therefore putting $\theta = \pi/2$ and 0 successively, and subtracting, we get

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n d\theta = \frac{(n-1)}{n}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{n-2} d\theta.$$

Similarly we have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{n-2}\, d\theta = \left(\frac{n-3}{n-2}\right)\int (\sin\theta)^{n-4}\, d\theta,$$

and so on, until, if n be even, we finally come to

$$\int_0^{\frac{1}{2}\pi} \sin^2\theta\, d\theta = \frac{1}{2}\cdot\frac{\pi}{2},$$

so that we thus obtain

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n\, d\theta = \frac{1\,.\,3\,.\,5\,\ldots\,n-1}{2\,.\,4\,.\,6\,\ldots\,n}\cdot\frac{\pi}{2}. \tag{7}$$

If n be odd, we come ultimately to

$$\int_0^{\frac{1}{2}\pi} \sin\theta\, d\theta = 1,$$

so that we then have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n\, d\theta = \frac{2\,.\,4\,.\,6\,\ldots\,n-1}{1\,.\,3\,.\,5\,\ldots\,n}. \tag{8}$$

It may be observed that these values of the definite integral might have been written down at once from (36) and (37), Art. 58.

Several other elementary definite integrals are immediately reducible to the preceding cases. For instance,

$$\int_0^a \frac{x^n\, dx}{\sqrt{(a^2-x^2)}} = a^n\int_0^{\frac{1}{2}\pi} (\sin\theta^n\, d\theta),$$

where $x = a\sin\theta$.

Also $$\int_0^\infty \frac{dx}{(a^2+x^2)^n} = \frac{1}{a^{2n-1}}\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2n-2}\, d\theta$$

$$= \frac{1}{a^{2n-1}}\,\frac{1\,.\,3\,.\,5\,\ldots\,2n-3}{2\,.\,4\,.\,6\,\ldots\,2n-2}\,\frac{\pi}{2}, \tag{9}$$

by putting $x = a\tan\theta$.

106. To find the value of

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n (\cos\theta)^m \, d\theta, \quad \text{or} \quad \int_0^{\frac{1}{2}\pi} (\sin\theta)^m (\cos\theta)^n \, d\theta,$$

where m and n are positive integers. From (45), Art. 62, we have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n (\cos\theta)^m \, d\theta = \left(\frac{n-1}{m+n}\right) \int_0^{\frac{1}{2}\pi} (\sin\theta)^{n-2} (\cos\theta)^m \, d\theta\,;$$

so that if n be odd, we find

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n (\cos\theta)^m \, d\theta = \frac{2\,.\,4\,.\,6\,\ldots\,n-1}{(m+3)(m+5)\ldots(m+n)} \int_0^{\frac{1}{2}\pi} \sin\theta\,(\cos\theta)^m \, d\theta$$

$$= \frac{2\,.\,4\,.\,6\,\ldots\,n-1}{(m+1)(m+3)\ldots(m+n)}. \qquad (10)$$

If both m and n be even, we have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^n (\cos\theta)^m \, d\theta = \frac{1\,.\,3\,.\,5\,\ldots\,n-1}{(m+2)(m+4)\ldots(m+n)} \int_0^{\frac{1}{2}\pi} (\cos\theta)^m \, d\theta$$

$$= \frac{1\,.\,3\,.\,5\,\ldots\,n-1\,.\,1\,.\,3\,.\,5\,\ldots\,m-1}{2\,.\,4\,.\,6\,\ldots\,(m+n)} \frac{\pi}{2} \qquad (11)$$

The definite integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} \, dx,$$

where m and n are positive, is reducible to the preceding; for, putting $x = \sin^2\theta$, it becomes

$$2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta,$$

which, from (10), is equal to

$$2\frac{2.4.6\ldots 2n-2}{(2m)(2m+2)\ldots(2m+2n-2)}$$

$$=\frac{1.2.3\ldots n-1}{m(m+2)\ldots(m+n-1)}, \qquad (12)$$

after dividing the numerator and denominator by 2^n.

This result evidently holds also when m is not an integer. When m is an integer, the expression (12) is symmetrical in m and n, for it may be then written

$$\frac{1.2.3\ldots n-1.1.2.3\ldots m-1}{1.2.3\ldots(m+n-1)}, \qquad (13)$$

as the value of the integral is easily seen to be in all cases by substituting $1-x$ for x.

Examples.

1. $$\int_0^1 (2ax-x^2)^{n+\frac{1}{2}}\,dx = a^{2n+2}\frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)}\frac{\pi}{2}.$$

2. $$\int_0^a (2ax-x^2)^n\,dx = a^{2n+1}\frac{2.4.6\ldots 2n}{1.3.5\ldots(2n+1)}.$$

3. $$\int_0^1 x^n(1-x^2)^m\,dx = \frac{2.4.6\ldots 2m}{(n+1)(n+3)\ldots(n+2m+1)}.$$

4. $$\int_{-\infty}^{\infty}\frac{dx}{(x^2-2x\cos a+1)^n} = \frac{\pi}{(\sin a)^{2n-1}}\frac{1.3.5\ldots 2n-3}{2.4.6\ldots 2n-2}.$$

5. Show that

$$\int_0^\infty \frac{x^{m-1}\,dx}{(a+bx)^{m+n}} = \frac{1}{a^n b^m}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

This result is obtained by putting $bx/(a+bx)$ for x in the integral on the right-hand side.

6. $$\int_0^\infty \frac{x^{2m-1}\,dx}{(1+x^2)^{m+\frac{1}{2}}} = \frac{2.4.6\ldots 2m-2}{1.3.5\ldots 2m-1}.$$

7. $$\int_0^{\frac{1}{2}\pi}\sin^6\theta\,d\theta = \frac{5\pi}{32}.$$

107. To find the value of $\int_0^\infty x^n e^{-x}\,dx$, where n is a positive integer. Taking $a = -1$ in (73), Art. 74, we have

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int_0^\infty x^{n-1} e^{-x}\,dx.$$

Hence, since $x^n e^{-x}$ vanishes both for $x = \infty$ and $x = 0$, we get

$$\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx = n(n-1)\int_0^\infty x^{n-2} e^{-x}\,dx = \&c., \qquad (14)$$

so that we obtain ultimately

$$\int_0^\infty x^n e^{-x}\,dx = 1\,.\,2\,.\,3\,\ldots\,n. \qquad (15)$$

The definite integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ has been called the second Eulerian Integral, and is usually denoted by the symbol Γ, that is, we write

$$\int_0^\infty x^{n-1} e^{-x}\,dx = \Gamma(n), \qquad (16)$$

where n is any positive quantity. Hence, from (14) we have

$$\Gamma(n+1) = n\Gamma(n), \qquad (17)$$

and from (15), if n is an integer,

$$\Gamma(n) = 1\,.\,2\,.\,3\,\ldots\,n-1. \qquad (18)$$

If we put ax for x in (16) we have, more generally,

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}}. \qquad (19)$$

Again, the integral

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx,$$

depends upon the function Γ, for putting $x = e^{-y}$ we get

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \int_0^\infty e^{-(m+1)y}\, y^n\, dy = \frac{\Gamma(n+1)}{(m+1)^{n+1}}. \quad (20)$$

In this integral we have written $\log (1/x)$ instead of $-\log x$, as the former of these quantities is always positive between the limits.

Taking $m = 0$, and writing $n - 1$ for n, we have, from (20),

$$\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n),$$

the form in which the function Γ was originally studied by Euler.

108. We have already explained in Art. 22 how we can derive several integrals from a given one by means of the process of differentiation or integration under the sign of integration.

This method is especially serviceable in the evaluation of definite integrals; and the formulae of Art. 22 will still apply in this case, provided of course that the limits of integration are independent of the quantity with respect to which we differentiate or integrate.

As an example of differentiation, let us consider the integral

$$\int_0^\infty e^{-ax} dx = \frac{1}{a}:$$

differentiating then n times with regard to a, we get

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{1.2.3\ldots n}{a^{n+1}},$$

as we have already shown.

Again, from the equation

$$\int_0^\infty \frac{dx}{x^2+c} = \frac{\pi}{2}\frac{1}{c^{\frac{1}{2}}},$$

we get, by differentiating $n-1$ times with regard to c,

$$\int_0^\infty \frac{dx}{(x^2+c)^n} = \frac{\pi}{2}\frac{1.3.5\ldots 2n-3}{2.4.6\ldots 2n-2}\frac{1}{c^{n-\frac{1}{2}}},$$

which agrees with (9).

109. By differentiation we can frequently make a definite integral depend upon a form in which the general integral can be obtained at once, so that then by integration we can determine the value of the proposed integral.

Thus, let
$$u = \int_0^\infty e^{-ax}\frac{\sin mx\,dx}{x};$$

then
$$\frac{du}{dm} = \int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2};$$

therefore
$$u = a\int\frac{dm}{a^2+m^2} = \tan^{-1}\left(\frac{m}{a}\right), \tag{21}$$

no constant being added as u vanishes with m. Or thus:

$$\frac{du}{da} = -\int_0^\infty e^{-ax}\sin mx\,dx = -\frac{m}{a^2+m^2};$$

therefore $u = -m\int\frac{da}{a^2+m^2} = \tan^{-1}\left(\frac{m}{a}\right)$, as before.

Again, let $$u = \int_0^1 \frac{(x^m - 1)\,dx}{\log x};$$

then $$\frac{du}{dm} = \int_0^1 \frac{x^m \log x\,dx}{\log x} = \int_0^1 x^m\,dx = \frac{1}{m+1};$$

therefore $$u = \log(m+1),$$

no constant being added, as u vanishes with m.

Examples.

1. To find $$u = \int_0^\pi \log(1 + n\cos\theta)\,d\theta.$$

We have $$\frac{du}{dn} = \int_0^\pi \frac{\cos\theta\,d\theta}{1 + n\cos\theta} = \frac{\pi}{n} - \frac{\pi}{n\sqrt{(1-n^2)}};$$

therefore $$u = \pi \log\left\{\frac{1 + \sqrt{(1-n^2)}}{2}\right\},$$

the constant being determined so that u may vanish with n.

2. $$\int_0^\infty \frac{\log(1 + a^2x^2)\,dx}{1 + x^2} = \pi\log(1+a).$$

3. $$\int_0^1 \tan^{-1}\left(\frac{2mx}{1+x^2}\right)\frac{dx}{x} = \frac{\pi}{2}\log\{m + \sqrt{(1+m^2)}\}.$$

4. $$\int_0^1 \frac{(x^{m-1} - x^{n-1})\,dx}{\log x} = \log\left(\frac{m}{n}\right).$$

5. $$\int_{-1}^1 \frac{\log(1+nx)\,dx}{1-x^2} = \pi\sin^{-1}n.$$

6. $$\int_0^1 \log(1 + 2mx + x^2)\frac{dx}{x} = \frac{\pi^2}{6} - \tfrac{1}{2}(\cos^{-1}m)^2,\ (m<1),$$

or $$\frac{\pi^2}{6} + \tfrac{1}{2}\left(\log\{m + \sqrt{(m^2-1)}\}\right)^2,\ (m>1).$$

7. $$\int_0^{\frac{1}{2}\pi} \log\left(\frac{1+n\cos\theta}{1-n\cos\theta}\right)\frac{d\theta}{\cos\theta} = \pi\sin^{-1}n.$$

8. $$\int_0^\infty (1-e^{-ax})\frac{\cos mx}{x}dx = \frac{1}{2}\log\left(1+\frac{a^2}{m^2}\right).$$

9. $$\int_0^\infty \tan^{-1}\left\{\frac{(a-1)x}{1+ax^2}\right\}\frac{dx}{x} = \frac{\pi}{2}\log a.$$

110. We now consider the important definite integral

$$I = \int_0^\infty e^{-x^2}dx.$$

Putting ax for x, we have

$$I = \int_0^\infty e^{-a^2x^2}a\,dx,$$

and
$$Ie^{-a^2}da = \int_0^\infty (e^{-a^2(1+x^2)}a\,da)dx.$$

Integrating now with regard to a between ∞ and 0, we have

$$I\int_0^\infty e^{-a^2}da = \int_0^\infty \frac{1}{2}\,\frac{1}{1+x^2}dx,$$

since
$$\int_0^\infty e^{-a^2(1+x^2)}a\,da = \frac{1}{2}\,\frac{1}{1+x^2};$$

therefore
$$I^2 = \frac{\pi}{4},$$

and
$$\int_0^\infty e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi}. \qquad (22)$$

By means of this result several other integrals can be obtained. Thus, putting $x\sqrt{a}$ for x, we have

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{a}},$$

which, being differentiated n times with regard to x, gives

$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{\sqrt{\pi}}{a^{n+\frac{1}{2}}} \frac{1.3.5\ldots(2n-1)}{2^{n+1}}.$$

But putting $(2n+1)/2$ for n in (16), we have

$$\Gamma(n+\tfrac{1}{2}) = \int_0^\infty x^{n-\frac{1}{2}} e^{-x} dx = 2\int_0^\infty x^{2n} e^{-x^2} dx,$$

by the substitution of x^2 for x; therefore

$$\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,\frac{1.3.5\ldots 2n-1}{2^n}, \tag{23}$$

and
$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}. \tag{24}$$

111. Let us consider the definite integral

$$u = \int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx.$$

We have
$$\frac{du}{da} = -2a\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} \frac{dx}{x^2};$$

but putting a/x for x in u, we get

$$u = a\int_0^\infty e^{-\left(\frac{a^2}{x^2}+x^2\right)} \frac{dx}{x^2};$$

therefore
$$\frac{du}{da} = -2u,$$

from which we get
$$u = Ce^{-2a}.$$

To determine C, let $a = 0$; then

$$C = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$$

Hence we have $$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2} e^{-2a}. \tag{25}$$

112. Again, to find

$$u = \int_0^\infty e^{-x^2} \cos 2mx\,dx.$$

We have $$\frac{du}{dm} = -2\int_0^\infty e^{-x^2} \sin 2mx\,x dx;$$

but, integrating by parts, we get

$$\int e^{-x^2} \sin 2mx\,x dx = -\tfrac{1}{2} e^{-x^2} \sin 2mx + m\int e^{-x^2} \cos 2mx\,dx;$$

hence $$\int_0^\infty e^{-x^2} \sin 2mx\,x dx = mu;$$

therefore $$\frac{du}{dm} = -2mu,$$

from which we obtain $u = Ce^{-m^2}$.

Putting, then, $m = 0$, we have

$$C = \sqrt{\pi}/2,$$

and $$\int_0^\infty e^{-x^2} \cos 2mx\,dx = \frac{\sqrt{\pi}}{2} e^{-m^2}. \tag{26}$$

If we put ax for x, and m/a for m, we have, more generally,

$$\int_0^\infty e^{-a^2x^2} \cos 2mx\,dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{m^2}{a^2}}. \tag{27}$$

113. We now consider the definite integral

$$u = \int_0^\infty \frac{\sin x}{x} dx.$$

The value of u is, from (2), the limiting value of the infinite series

$$\sin h + \tfrac{1}{2} \sin 2h + \tfrac{1}{3} \sin 3h + \&c.,$$

when h is indefinitely diminished. Now the sum of this series between 2π and small values of h is $(\pi - h)/2$; so that letting $h = 0$, we have $u = \pi/2$. Also, putting cx for x, we get

$$\int_0^\infty \frac{\sin cx}{x} dx = \frac{\pi}{2}. \tag{28}$$

This integral might have been obtained at once from (21) by putting $a = 0$; but it is to be observed that (21) is determined by means of a definite integral, in which a is expressly supposed to have a finite value, so that the method of evaluation just given seems more rigorous.

114. This integral will assist us to determine some others. Thus, writing

$$u = \int_0^\infty \frac{x \sin cx\, dx}{1 + x^2},$$

we have
$$u - \frac{\pi}{2} = \int_0^\infty \frac{x \sin cx\, dx}{1 + x^2} - \int_0^\infty \frac{\sin cx}{x} dx$$

$$= - \int_0^\infty \frac{\sin cx\, dx}{x(1 + x^2)}; \tag{29}$$

hence, differentiating both sides twice with regard to c, we have

$$\frac{d^2u}{dc^2} = \int_0^\infty \frac{x \sin cx\, dx}{1 + x^2} = u.$$

Multiplying both sides by du/dc, we have

$$\frac{d^2u}{dc^2}\frac{du}{dc} = \frac{u\,du}{dc},$$

or
$$\frac{1}{2}\frac{d}{dc}\left(\frac{du^2}{dc^2}\right) = \frac{1}{2}\frac{d(u^2)}{dc};$$

therefore
$$\left(\frac{du}{dc}\right)^2 = u^2 + a,$$

and
$$\frac{du}{\sqrt{(u^2+a)}} = dc.$$

Hence, by integration,

$$\log\{u + \sqrt{(u^2 + a)}\} = c + \beta,$$

or
$$u + \sqrt{(u^2 + a)} = e^{(c+\beta)};$$

also
$$\sqrt{(u^2 + a)} - u = ae^{-(c+\beta)},$$

from which we get

$$2u = e^{(c+\beta)} - ae^{-(c+\beta)},$$

or, as it may be written,

$$u = Ce^{-c} + C'e^{c},$$

where C, C' are constants. Now u evidently does not increase indefinitely with c; therefore $C' = 0$. Also, let c be very small in (29), and we have

$$u - \frac{\pi}{2} \doteq -\int_0^\infty \frac{c\,dx}{1+x^2} = -\frac{c\pi}{2} = 0,$$

when c vanishes. We thus get $C = \pi/2$, and

$$u = \int_0^\infty \frac{x \sin cx\,dx}{1+x^2} = \frac{\pi}{2}e^{-c}, \tag{30}$$

where c is supposed to be essentially positive. We have, also,

$$\int_0^\infty \frac{\sin cx\,dx}{x(1+x^2)} = \frac{\pi}{2} - u = \frac{\pi}{2}(1 - e^{-c});$$

therefore, differentiating with regard to c, we get

$$\int_0^\infty \frac{\cos cx\,dx}{1+x^2} = \frac{\pi}{2}e^{-c}. \tag{31}$$

115. A considerable number of definite integrals can be obtained by expanding the expression to be integrated in a convergent series, integrating each term separately, and then summing the resulting series.

For example, to evaluate the integral

$$u = \int_0^1 \frac{\log\left(\frac{1}{x}\right)dx}{1-x},$$

we expand $1/(1-x)$ in the form

$$1 + x + x^2 + \&c.$$

Observing, then, that

$$\int_0^1 x^m \log\left(\frac{1}{x}\right)dx = \frac{1}{(m+1)^2},$$

we find

$$u = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c.,$$

which, by a known result in Trigonometry, is equal to $\pi^2/6$.

Again, considering the expansion of

$$\log(1 + m^2 + 2m\cos\theta),$$

we have $\log(1+m^2+2m\cos\theta) = \log(1+me^{i\theta}) + \log(1+me^{-i\theta})$

$$= m(e^{i\theta}+e^{-i\theta}) - \tfrac{1}{2}m^2(e^{2i\theta}+e^{-2i\theta}) + \&c.$$

$$= 2(m\cos\theta - \tfrac{1}{2}m^2\cos 2\theta + \tfrac{1}{3}m^3\cos 3\theta + \&c.), \qquad (32)$$

if m is less than unity. Hence we have

$$\int_0^\pi \log(1+m^2+2m\cos\theta)\,d\theta = 0, \qquad (33)$$

when m is < 1; but if m is > 1, we have

$$\log(1+m^2+2m\cos\theta) = 2\log m + \log(1+m^{-2}+2m^{-1}\cos\theta)$$

$$= 2\log m + 2(m^{-1}\cos\theta - \tfrac{1}{2}m^{-2}\cos 2\theta + \tfrac{1}{3}m^{-3}\cos 3\theta + \&c.);$$

therefore $$\int_0^\pi \log(1+m^2+2m\cos\theta)\,d\theta = 2\pi\log m. \qquad (34)$$

Hence, also, we have

$$\int_0^\pi \log(a+b\cos\theta)\,d\theta = \pi\log\left\{\frac{a+\sqrt{(a^2-b^2)}}{2}\right\}. \qquad (35)$$

Again, multiplying both sides of (32) by $\cos r\theta$, we find

$$\int_0^\pi \cos r\theta \log(1+m^2+2m\cos\theta)\,d\theta = -\frac{\pi}{r}(-m)^r, \text{ or } -\frac{\pi}{r}(-m)^{-r},$$

according as m is less than or greater than unity; for, by Ex. 13, Art. 103, every term vanishes at the limits except that involving $\cos^2 r\theta$.

This latter result is evidently a particular case of the general theorem, that if it be possible to expand $f(\theta)$ in an infinite converging series of the form

$$a_0 + a_1\cos\theta + a_2\cos 2\theta + \&c.,$$

then $$\int_0^\pi \cos r\theta f(\theta)\,d\theta = \frac{\pi a_n}{2}, \qquad (36)$$

for all values of r greater than zero, provided $f(\theta)$ retains the same form between the limits.

116. Several definite integrals of the form

$$\int_0^\infty \frac{f(x)\,dx}{1+x^2}$$

can be deduced from (31), in cases in which it is possible to expand $f(x)$ in a series of cosines of the form $\cos rcx$, provided of course that $f(x)$ is continuous, and retains the same form for all values of x between the limits.

For example, if m is < 1, we have

$$\log(1+m^2+2m\cos cx)$$
$$= 2\left(m\cos cx - \tfrac{1}{2}m^2\cos 2cx + \tfrac{1}{3}m^3\cos 3cx + \&c.\right);$$

therefore

$$\int_0^\infty \log(1+m^2+2m\cos cx)\frac{dx}{1+x^2} = \pi\left(me^{-c} - \tfrac{1}{2}m^2e^{-2c} + \tfrac{1}{3}m^3e^{-3c} + \&c.\right)$$
$$= \pi\log(1+me^{-c}). \qquad (37)$$

Also, if m is > 1, we find

$$\int_0^\infty \log(1+m^2+2m\cos cx)\frac{dx}{1+x^2} = \pi\log(m+e^{-c}). \qquad (38)$$

Hence, if we make $m = 1$ in either of these results, we get

$$\int_0^\infty \log(\cos^2\tfrac{1}{2}cx)\frac{dx}{1+x^2} = \pi\log\left(\frac{1+e^{-c}}{2}\right), \qquad (39)$$

in which it is evident that we ought not to write

$$2\log\cos(cx/2) \text{ for } \log\cos^2(cx/2);$$

for the former quantity assumes imaginary values an infinite number of times between the limits.

EXAMPLES.

1. Show that

$$\int_0^\infty \frac{\sin ax \cos bx}{x} dx = \frac{\pi}{2}, \frac{\pi}{4}, \text{ or } 0,$$

according as a is $>$, $=$, or $< b$.

2. $\displaystyle\int_0^\infty \sin^2 ax \frac{dx}{x^2} = \frac{\pi a}{2}.$

3. $\displaystyle\int_0^1 \frac{\log\left(\frac{1}{x}\right) dx}{1+x} = \frac{\pi^2}{12}.$

4. $\displaystyle\int_0^\pi \frac{\theta \sin\theta\, d\theta}{1+m^2+2m\cos\theta} = \frac{\pi}{m}\log\frac{1}{1-m}\ (m^2<1), \text{ or } \frac{\pi}{m}\log\frac{m}{m-1}\ (m^2>1).$

This result can be obtained by integrating (33) and (34) by parts.

5. $\displaystyle\int_0^\pi \frac{\sin\theta \sin r\theta\, d\theta}{1+m^2-2m\cos\theta} = \frac{\pi}{2} m^{r-1}, \text{ or } \frac{\pi}{2} m^{-(r+1)},$ according as m^2 is $<$ or > 1.

6. $\displaystyle\int_0^\pi \frac{\cos r\theta\, d\theta}{1+m^2-2m\cos\theta} = \frac{\pi m^r}{1-m^2}\ (m^2<1).$

7. $\displaystyle\int_0^\pi (\cos\theta)^n \cos r\theta\, d\theta = 0$, if r is $> n$, and also if $n-r$ is odd; but if $n-r$ is even $= 2i$, say,

$$\int_0^\pi (\cos\theta)^n \cos r\theta\, d\theta = \frac{\pi}{2^n}\frac{n(n-1)\ldots(n-i+1)}{1.2.3\ldots i}.$$

8. $\displaystyle\int_0^\infty \frac{x \sin cx\, dx}{(1+x^2)(1+m^2+2m\cos cx)} = \frac{\pi}{2}\frac{1}{m+e^c}\ (m^2<1).$

9. $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)(1+m^2+2m\cos cx)} = \frac{\pi(1-me^{-c})}{2(1-m^2)(1+me^{-c})}\ (m^2<1).$

10. $\displaystyle\int_0^\infty \frac{x \tan cx\, dx}{1+x^2} = \frac{\pi}{1+e^{2c}}.$

11. $\displaystyle\int_0^\infty \frac{x\, dx}{\sin cx\,(1+x^2)} = \frac{\pi}{e^c-e^{-c}}.$

117. From given definite integrals we can frequently deduce several others by the use of imaginaries. For instance, putting $a + i\beta$ for a in

$$\int_0^\infty x^{n-1} e^{-ax} dx = \frac{\Gamma(n)}{a^n},$$

and equating the real and imaginary parts on both sides of the resulting equation, we get

$$\left.\begin{aligned} \int_0^\infty x^{n-1} e^{-ax} \cos \beta x \, dx &= \frac{\Gamma(n)}{\gamma^n} \cos n\theta, \\ \int_0^\infty x^{n-1} e^{-ax} \sin \beta x \, dx &= \frac{\Gamma(n)}{\gamma^n} \sin n\theta, \end{aligned}\right\} \tag{40}$$

where on the right-hand side we have put $\gamma \cos \theta$, $\gamma \sin \theta$ for a, β, respectively.

If we suppose a to vanish in these results, we have $\theta = \pi / 2$, and we get

$$\left.\begin{aligned} \int_0^\infty x^{n-1} \cos \beta x \, dx &= \frac{\Gamma(n)}{\beta^n} \cos \frac{n\pi}{2}, \\ \int_0^\infty x^{n-1} \sin \beta x \, dx &= \frac{\Gamma(n)}{\beta^n} \sin \frac{n\pi}{2}. \end{aligned}\right\} \tag{41}$$

The method by which these integrals have been arrived at does not seem rigorous; but they can be obtained otherwise, as we shall show further on. It is to be observed that, in (41), n must be supposed to lie between the limits unity and zero.

Again, from (27), by changing m into $(im)/2$, we have

$$\int_0^\infty e^{-a^2 x^2} (e^{mx} + e^{-mx}) dx = \frac{\sqrt{\pi}}{a} e^{\frac{m^2}{4a^2}}. \tag{42}$$

EXAMPLES.

1. $$\int_0^\infty e^{-\alpha x^2}\cos\beta x^2\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\gamma}}\cos\frac{\theta}{2},$$

$$\int_0^\infty e^{-\alpha x^2}\sin\beta x^2\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\gamma}}\sin\frac{\theta}{2},$$

where $$\alpha = \gamma\cos\theta,\quad \beta = \gamma\sin\theta.$$

2. $$\int_0^\infty e^{-u\cos\alpha}\cos(u\sin\alpha)\,dx = \frac{\sqrt{\pi}}{2}e^{-2a\cos\alpha}\cos\beta,$$

$$\int_0^\infty e^{-u\cos\alpha}\sin(u\sin\alpha)\,dx = \frac{\sqrt{\pi}}{2}e^{-2a\cos\alpha}\sin\beta,$$

where $$u = (x^2 + a^2/x^2)\cos\alpha,\quad \beta = 2a\sin\alpha + \alpha/2.$$

3. $$\int_0^\infty \cos mx^2\cos(cx-\theta)\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{m}}\cos\left(\frac{\pi}{4} - \frac{c^2}{4m} - \theta\right).$$

4. $$\int_0^\infty \frac{e^{a\cos cx}\cos(a\sin cx)\,dx}{1+x^2} = \frac{\pi}{2}e^{ae^{-c}}.$$

118. We now proceed to consider the definite integral

$$\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}},$$

where m and n are positive integers, and $n > m$. We have

$$\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \int_0^\infty \frac{x^{2(n-m-1)}}{1+x^{2n}}dx,$$

by putting $1/x$ for x; also,

$$\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^{2m}\,dx}{1+x^{2n}},$$

so that we get

$$\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \frac{1}{4}\int_{-\infty}^{\infty} \frac{(x^{2m}+x^{2m'})}{1+x^{2n}},$$

where $$m' = n - m - 1.$$

Now, resolving $$x^{2m}/(1+x^{2n})$$

into partial fractions, by Art. 27, the general term is

$$\frac{a^{2m}}{2na^{2n-1}(x-a)} = -\frac{a^{2m+1}}{2n(x-a)},$$

where a is a root of $1 + x^{2n} = 0$; that is, by the theory of equations, a is of the form

$$e^{i\theta}, \text{ where } \theta = (2r+1)\pi/2n.$$

Hence, taking together the two roots corresponding to a and a^{-1}, we have

$$-\frac{a^{2m+1}}{2n(x-a)} - \frac{a^{-(2m+1)}}{2n(x-a^{-1})} = -\frac{1}{2n}\left\{\frac{x\{a^{2m+1}+a^{-(2m+1)}\}-(a^{2m}+a^{-2m})}{x^2 - xa + a^{-1} + 1}\right\}$$

$$= -\frac{1}{n}\,\frac{x\cos(2m+1)\theta - \cos 2m\theta}{x^2 - 2x\cos\theta + 1};$$

and in the same way, in the resolution of $x^{2m'}/(1+x^{2n})$, we get the term

$$-\frac{1}{n}\,\frac{x\cos(2m'+1)\theta - \cos 2m'\theta}{x^2 - 2x\cos\theta + 1}.$$

Now,

$$\cos(2m'+1)\theta = \cos(2n-2m-1)\theta = -\cos(2m+1)\theta,$$

and

$$\cos 2m'\theta = \cos(2n-2m-2)\theta = -\cos(2m+2)\theta.$$

Hence, adding the two expressions together, we have

$$\frac{2 \sin \theta \sin (2m+1) \theta}{n (x^2 - 2x \cos \theta + 1)},$$

which, being multiplied by dx, and integrated between the limits, positive and negative infinity, becomes

$$\frac{2 \sin \theta \sin (2m+1) \theta}{n} \frac{\pi}{\sin \theta} = \frac{2\pi}{n} \sin (2m+1) \theta.$$

Hence

$$\int_{-\infty}^{\infty} \frac{(x^{2m} + x^{2m'})\, dx}{1 + x^{2n}} = \frac{2\pi}{n} \{\sin \phi + \sin 3\phi + \sin 5\phi + \ldots + \sin (2n-1) \phi\},$$

where $\phi = (2m+1)\pi / 2n.$

To find the sum of this series, let

$$S = \sin \phi + \sin 3\phi + \ldots + \sin (2n-1) \phi;$$

then

$$\begin{aligned} 2S \sin \phi &= 1 - \cos 2\phi + \cos 2\phi - \cos 4\phi + \ldots \\ &\qquad + \cos (2n-2) \phi - \cos 2n\phi \\ &= 1 - \cos 2n\phi = 2 \sin^2 n\phi = 2 \sin^2 (2m+1) \frac{\pi}{2} = 2. \end{aligned}$$

Thus,

$$S = \frac{1}{\sin \phi} = \frac{1}{\sin \dfrac{(2m+1)\pi}{2n}},$$

and

$$\int_0^{\infty} \frac{x^{2m}\, dx}{1 + x^{2n}} = \frac{1}{4} \int_0^{\infty} \frac{(x^{2m} + x^{2 \cdot (n-m-1)})\, dx}{1 + x^{2n}} = \frac{\pi}{2n \sin \dfrac{(2m+1)\pi}{2n}}. \quad (43)$$

Putting $x^{2n} = z$, and a for $(2m+1)/2n$ in the preceding result, we get

$$\int_0^\infty \frac{z^{a-1}\,dz}{1+z} = \frac{\pi}{\sin a\pi}. \tag{44}$$

Since (43) holds for all integer values of m and n, n being $> m$, it follows by the law of continuity that (44) holds for all values of a between zero and unity.

If we put now x^2 for z, and n for $2a-1$ in (44), we have

$$\int_0^\infty \frac{x^n\,dx}{1+x^2} = \frac{\pi}{2\cos\dfrac{n\pi}{2}}, \tag{45}$$

where n lies between the limits ± 1.

Again, we have

$$\int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \int_1^\infty \frac{x^{2m}\,dx}{1+x^{2n}} + \int_0^1 \frac{x^{2m}\,dx}{1+x^{2n}};$$

but putting $1/x$ for x in the first integral on the right-hand side, it becomes

$$\int_0^1 \frac{x^{2(n-m-1)}\,dx}{1+x^{2n}};$$

so that we get thus

$$\int_0^1 \frac{\left(x^{2m}+x^{2(n-m-1)}\right)dx}{1+x^{2n}} = \int_0^\infty \frac{x^{2m}\,dx}{1+x^{2n}} = \frac{\pi}{2n\sin\dfrac{(2m+1)\pi}{2n}}. \tag{46}$$

119. To find the value of

$$\int_0^1 \frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1-x^n},$$

where n is $> m$.

In this case the required result can be most readily obtained by the method of expansion in an infinite series. Substituting

$$1 + x^n + x^{2n} + \&c. \quad \text{for} \quad 1/(1 - x^n),$$

the proposed integral becomes

$$\int_0^1 (x^{m-1} - x^{n-m-1})(1 + x^n + x^{2n} + \&c.)$$

$$= \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \&c.$$

$$= \frac{1}{m} - \frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} - \&c.$$

To sum this series, let us consider the known expression for $\sin \pi z$ in an infinite product, namely,

$$\sin \pi z = \pi z (1 - z^2)(1 - z^2/2^2)(1 - z^2/3^2) \ \&c.;$$

taking then the differential coefficient of the logarithm of each side, we get

$$\pi \cot \pi z = \frac{1}{z} - \frac{2z}{1 - z^2} - \frac{2z}{4 - z^2} - \&c.,$$

which, by the substitution of m/n for z, gives

$$\frac{\pi}{n} \cot \frac{m\pi}{n} = \frac{1}{m} - \frac{2m}{n^2 - m^2} - \frac{2m}{4n^2 - m^2} - \&c.;$$

so that by the expression found above we have

$$\int_0^1 \frac{(x^{m-1} - x^{n-m-1})dx}{1 - x^n} = \frac{\pi}{n} \cot \frac{m\pi}{n}. \qquad (47)$$

If in this result we take $n = 2$, and put $1 + a$ for m, we get

$$\int_0^1 \frac{(x^a - x^{-a})}{x - x^{-1}} \frac{dx}{x} = \tan\frac{\pi a}{2}, \tag{48}$$

where a lies between the limits ± 1.

120. Several other integrals can be obtained from those given in the preceding Articles by means of the methods already explained. Thus, for example, differentiating both sides of (44) with regard to a, we have

$$\int_0^\infty \frac{x^{a-1}\log\left(\frac{1}{x}\right) dx}{1+x} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}. \tag{49}$$

Also integrating (46) with regard to m,

$$\int_0^1 \frac{(x^{2m} - x^{2(n-m-1)})\, dx}{(1 + x^{2n}) \log x} = \log \tan \frac{(2m+1)\pi}{4n}. \tag{50}$$

Again, putting x for x^n, and a for $(2m + 1 - n)/n$ in (46), we have

$$\int_0^1 \frac{(x^a + x^{-a})\, dx}{(x - x^{-1})\, x} = \frac{\pi}{2} \sec \frac{\pi a}{2}, \tag{51}$$

which, by the substitution of $e^{-\pi z}$ for x, and a for πa, gives

$$\int_0^\infty \frac{(e^{az} + e^{-az})\, dz}{e^{\pi z} + e^{-\pi z}} = \frac{1}{2} \sec \frac{a}{2}, \tag{52}$$

where a lies between the limits $\pm \pi$.

Similarly, from (47), we get

$$\int_0^\infty \frac{(e^{az} - e^{-az})\, dz}{e^{\pi z} - e^{-\pi z}} = \frac{1}{2} \tan \frac{a}{2}. \tag{53}$$

EXAMPLES.

1. $\int_0^1 \frac{(1+x^4)\,dx}{1+x^6} = \frac{\pi}{3}.$

2. $\int_0^1 \frac{x^{m-1}\,dx}{(1+ax)(1-x)^m} = \frac{\pi}{(1+a)^m \sin m\pi}.$

This may be obtained from (44) by putting $x/(1+a)(1-x)$ for x.

3. $\int_0^{\frac{1}{2}\pi} (\tan\theta)^{2m-1}\,d\theta = \frac{\pi}{2\sin m\pi},$ where $m > 0$ and < 1.

4. $\int_0^{\frac{1}{2}\pi} \{(\tan\theta)^n + (\cot\theta)^n\}\,d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2},$ where $n^2 < 1$.

5. $$\int_0 \frac{(e^{ax} + e^{-ax})}{e^{\pi x} + e^{-\pi x}} \cos\beta x\,dx = \frac{(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}})\cos\frac{\alpha}{2}}{e^{\beta} + e^{-\beta} + 2\cos\alpha}.$$

$$\int_0^{\infty} \frac{(e^{ax} - e^{-ax})}{e^{\pi x} + e^{-\pi x}} \sin\beta x\,dx = \frac{(e^{\frac{\beta}{2}} - e^{-\frac{\beta}{2}})\sin\frac{\alpha}{2}}{e^{\beta} + e^{-\beta} + 2\cos\alpha}, \quad \text{where } \alpha < \pi.$$

These integrals are obtained by putting $a + i\beta$ for a in (52).

6. $$\int_0^{\infty} \frac{(e^{ax} - e^{-ax})}{e^{\pi x} - e^{-\pi x}} \cos\beta x\,dx = \frac{\sin\alpha}{e^{\beta} + e^{-\beta} + 2\cos\alpha},$$

$$\int_0^{\infty} \frac{(e^{ax} + e^{-ax})}{e^{\pi x} - e^{-\pi x}} \sin\beta x\,dx = \frac{1}{2}\,\frac{e^{\beta} - e^{-\beta}}{e^{\beta} + e^{-\beta} + 2\cos\alpha}.$$

7. $$\int_0^{\infty} \frac{(e^{ax} - e^{-ax})}{e^{\pi x} - e^{-\pi x}}\,\frac{\sin\beta x\,dx}{x} = \tan^{-1}\left\{\frac{e^{\beta} - 1}{e^{\beta} + 1}\tan\frac{\alpha}{2}\right\}.$$

8. $$\int_0^{\infty} \frac{(e^{ax} - e^{-ax})}{e^{\pi x} + e^{-\pi x}}\,\frac{\cos\beta x\,dx}{x} = \frac{1}{2}\log\left\{\frac{e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}} + 2\sin\frac{\alpha}{2}}{e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}} - 2\sin\frac{\alpha}{2}}\right\}.$$

121. We have already explained and exemplified the process of differentiation under the sign of integration. It is necessary, however, to consider the case in which the limits are functions of the quantity with regard to which we differentiate.

Let the indefinite integral of $f(x, \alpha)\,dx$ be denoted by $\phi(x, \alpha)$; then we have

$$u = \int_b^a f(x, \alpha)\,dx = \phi(a, \alpha) - \phi(b, \alpha);$$

therefore $$\frac{du}{da} = \frac{d\phi(a, \alpha)}{da} = f(a, \alpha),$$

and $$\frac{du}{db} = -f(b, \alpha);$$

also $$\left(\frac{du}{d\alpha}\right) = \frac{d\phi(a, \alpha)}{d\alpha} - \frac{d\phi(b, \alpha)}{d\alpha} = \int_b^a \frac{df(x, \alpha)}{d\alpha}\,dx,$$

where the differentiation is partial with regard to α. Hence

$$\frac{du}{d\alpha} = \left(\frac{du}{d\alpha}\right) + \frac{du}{da}\frac{da}{d\alpha} + \frac{du}{db}\frac{db}{d\alpha}$$

$$= \int_b^a \frac{df(x, \alpha)}{d\alpha}\,dx + f(a, \alpha)\frac{da}{d\alpha} - f(b, \alpha)\frac{db}{d\alpha}, \quad (54)$$

which is the required formula in the case under consideration.

Hence, it may be observed, if a, b, are values of x which make $f(x, \alpha)$ vanish, we may differentiate as if a, b were independent of α.

122. Among the definite integrals considered in this Chapter, there are several in which one or both of the limits are infinite, and also some for which $f(x)$ in $\int_b^a f(x)dx$ becomes

infinite at or between the limits a, b. In the first case there is no difficulty, for we can make the limits finite by the substitution of a new variable. The latter case we propose to consider more particularly.

It is to be observed that we have invariably supposed in each integral that all the increments remain indefinitely small between the limits. If this were not so, that is, if one of the increments had a finite or infinite value, it is evident that the value of the integral would be infinite; in the former case, because there would be an infinite number of such increments; for, if an increment had a finite value, the consecutive increment also should be finite. It might be considered, however, in some cases that a positive and negative infinity would neutralize one another, and that thus the integral would have a finite value; but it is to be observed, that then, in fact, the value of the integral would be indeterminate, as the difference of two infinities may have any assignable value whatever. Writing the increment in the form $hf(x)$, where h is the interval, this could have a finite or infinite value only when $f(x) = \infty$. Let a be the value or one of the values of x which satisfies this equation, then the increment corresponding to the value $a + h$ of x will be $hf(a + h)$, which takes an indeterminate form when h is indefinitely diminished. We see thus that if a lie between the limits, or coincides with one of them, the integral will contain an increment which takes an indeterminate form; and this indeterminate expression, as we have seen, must vanish in the limit if the integral is to have a finite value. To express this condition put $f(x) = 1/\phi(x)$; then since $f(a) = \infty$, we have $\phi(a) = 0$, and the increment, $h/\phi(a + h)$, by the Differential Calculus, becomes $1/\phi'(a)$ in the limit; and as this must vanish, we get $\phi'(a) = \infty$.

For instance, if $f(x)$ is of the form

$$A(x-a)^{-n} + B\log(x-a),$$

where n is a positive quantity, and A, B are functions of x, which remain finite when $x = a$, the increment corresponding to $x = a + h$ is

$$Ah^{1-n} + Bh\log h,$$

which vanishes in the limit, provided $n < 1$.

123. In connection with the question of finding the approximate values of definite integrals, the expansion of an integral in an infinite series becomes of importance. We do not propose here to enter into any details on the subject; but merely give the fundamental formula, namely, Bernouilli's series, which may be obtained as follows:—Integrating by parts, we have—

$$\int_0^x f(z)dz = xf(x) - \int_0^x zf'(z)dz,$$

$$\int_0^x zf'(z)dz = \frac{x^2 f'(x)}{1 \,.\, 2} - \int \frac{z^2}{1 \,.\, 2} f''(z)dz;$$

therefore

$$\int_0^x f(z)dz = xf(x) - \frac{x^2}{1 \,.\, 2} f'(x) + \int_0^x \frac{z^2}{1 \,.\, 2} f''(z).$$

Proceeding in this manner, we find

$$\int_0^x f(z)dz = xf(x) - \frac{x^2}{1 \,.\, 2} f'(x) + \frac{x^3}{1 \,.\, 2 \,.\, 3} f''(x) - \&c., \quad (55)$$

the remainder after n terms being expressed in the form of the definite integral

$$(-1)^n \int_0^x \frac{z^n f^{(n)}(z)dz}{1 \,.\, 2 \,.\, 3 \ldots n}.$$

124. We may notice here an expansion for the complete elliptic integral of the first kind.

Putting $n = (1-k')/(1+k')$, we have $k' = (1-n)/(1+n)$, and $k^2 = 4n/(1+n^2)$; therefore

$$\Delta(\theta) = \sqrt{(1-k^2\sin^2\theta)} = \sqrt{(1+n^2+2n\cos 2\theta)}/(1+n)$$

$$= \sqrt{\{(1+ne^{2i\theta})(1+ne^{-2i\theta})\}}/(1+n).$$

Hence $$K = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\Delta(\theta)} = (1+n)\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{\{(1+ne^{2i\theta})(1+ne^{-2i\theta})\}}}$$

$$= (1+n)\int_0^{\frac{1}{2}\pi} \left\{1 - \frac{1}{2}ne^{2i\theta} + \frac{1.3}{2.4}e^{4i\theta} - \&c.\right\}$$

$$\times \left\{1 - \frac{1}{2}ne^{-2i\theta} + \frac{1.3}{2.4}e^{-4i\theta} - \&c.\right\}d\theta$$

$$= (1+n)\int_0^{\frac{1}{2}\pi} \left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.3}{2.4}\right)^2 n^4 + \&c.\right.$$

$$\left. + 2A\cos 2\theta + 2B\cos 4\theta + \&c.\right\}d\theta.$$

Hence we have

$$K = \frac{\pi}{2}(1+n)\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1.3}{2.4}\right)^2 n^4 + \&c.\right\}. \quad (56)$$

EXAMPLES.

1. Given $$f(x+h) - f(x) = \int_0^h f'(x+h-z)\,dz,$$

deduce Taylor's series by means of the method of integration by parts, and hence express the remainder after n terms by a definite integral.

2. Show that the series (56) can be readily obtained by Landen's transformation (see Art. 94).

3. Show that

$$E = \frac{\pi}{2(1+n)}\left\{1 + \left(\frac{1}{2}\right)^2 n^2 + \left(\frac{1\,.\,1}{2\,.\,4}\right)^2 n^4 + \left(\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}\right)^2 n^6 + \&c.,\right.$$

where $n = (1 - k')/(1 + k')$.

4. $$\int_x^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} - \frac{e^{-x^2}}{2x}\left\{1 - \frac{1}{2}x^{-2} + \frac{1\,.\,3}{2^2}x^{-4} - \frac{1\,.\,3\,.\,5}{2^3}x^{-6} + \&c.\right\}$$

This series is always divergent, but will serve to give approximate values of the integral for large values of x.

5. If $x^2 < 1$, show that

$$\int_0^x \frac{dx}{\sqrt{(1+x^4)}} = x - \frac{1}{2}\frac{x^5}{5} + \frac{1\,.\,3}{2\,.\,4}\frac{x^9}{9} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^{13}}{13} + \&c.$$

125. The definite integrals considered in this chapter are those which most obviously suggest themselves in analytical processes. In geometrical and physical investigations, however, there occur a large number of definite integrals, which can be evaluated by methods directly suggested by considerations of these branches of knowledge. This is especially the case with integrals which occur in the theory of attractions.

As an example of the use of geometrical methods, let us consider the integral

$$I = \int_0^{2\pi} \log\{(x - a\cos\theta)^2 + (y - b\sin\theta)^2\}\,d\theta,$$

where $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = U < 0.$$

Let x, y be the rectangular co-ordinates of a point, then $a\cos\theta$, $b\sin\theta$, are the co-ordinates of a point on the ellipse $U = 0$, and we may write

$$I = \int \log(R^2)\,d\theta,$$

where R is the distance of a point P on U from a point O inside U, and the integration is taken through the entire perimeter of the curve. Now, differentiating with regard to x, we have

$$\frac{dI}{dx} = 2\int_0^{2\pi} \frac{(x - a\cos\theta)d\theta}{(x - a\cos\theta)^2 + (y - b\sin\theta)^2};$$

and we propose to show that this integral is equal to zero. Putting

$$x - a\cos\theta = R\cos\psi, \quad y - b\sin\theta = R\sin\psi,$$

we get

$$\frac{dI}{dx} = 2\int_0^{2\pi} \frac{\cos\psi\, d\theta}{R} = \int_0^{2\pi} \frac{\cos\psi\, d\theta'}{R'},$$

where θ', R' correspond to the point Q, in which the line OP

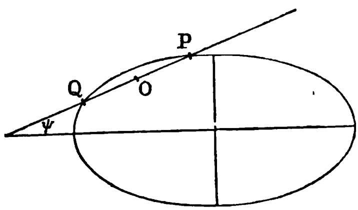

Fig. 3.

meets the curve again. Hence

$$\frac{dI}{dx} = \int_0^{2\pi} \cos\psi \left(\frac{d\theta}{R} + \frac{d\theta'}{R'}\right).$$

But $d\theta/R + d\theta'/R'$ vanishes; for projecting the ellipse orthogonally into a circle, $d\theta$, $d\theta'$ being the differentials of the eccentric angles of points on the ellipse, become proportional to the elements of the arcs at the corresponding points

on the circle, and the ratio R/R' remains unaltered. The relation then follows by considerations of elementary geometry.

We have thus proved that I is independent of x, and therefore of y. Putting then both of these quantities equal to zero, we find, making use of (35),

$$I = \int_0^{2\pi} \log(a^2 \cos^2\theta + b^2 \sin^2\theta) d\theta = 4\pi \log\left(\frac{a+b}{2}\right). \tag{57}$$

EXAMPLES.

1. $\displaystyle\int_0^{2\pi} \log\{(x - a\cos\theta)^2 + (y - b\sin\theta)^2\} d\theta = 4\pi \log\left\{\frac{\sqrt{(\lambda^2+a^2)}+\sqrt{(\lambda^2+b^2)}}{2}\right\}$,

where
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 > 0,$$

and λ^2 is the greatest root of the equation

$$\frac{x^2}{\lambda^2 + a^2} + \frac{y^2}{\lambda^2 + b^2} - 1 = 0.$$

2. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(x - a\cos\theta)^2 + (y - b\sin\theta)^2} = \frac{2\pi ab}{a^2 b^2 - b^2 x^2 - a^2 y^2}$,

where
$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} > 0.$$

3. $\displaystyle\int_0^{2\pi} \log(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta + 2g\cos\theta + 2f\sin\theta + c)\, d\theta$

$$= 2\pi \log\left(\frac{a + b + 2\lambda + 2\sqrt{\{(\lambda+a)(\lambda+b) - h^2\}}}{4}\right),$$

where λ is a properly selected root of the cubic equation

$$\begin{vmatrix} a+\lambda, & h, & g \\ h, & b+\lambda, & f \\ g, & f, & c-\lambda \end{vmatrix} = 0.$$

126. This chapter would be incomplete without some brief account of the properties of the Gamma function. This function is of great importance in the theory of definite integrals, as by its use we are enabled to express a large number of them in a known form. For this reason tables of $\log \Gamma(p)$ have been constructed by Legendre for values of p between 1 and 2. When these values are given, those corresponding to any other value of p can be found by means of the relation

$$\Gamma(p+1) = p\Gamma(p).$$

The most usual definition of the function Γ is by means of the equation (16), namely,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx,$$

from which we derive (17),

$$\Gamma(n+1) = n\Gamma(n),$$

which is true for all values of n. It may be observed, however, that the definite integral itself does not give an absolutely correct definition; for the integral becomes infinite for every negative value of n; but this could not be the case for any continuous function of a variable. To exemplify this, let $n = -1/2$; then from (17) we get

$$\Gamma(-\tfrac{1}{2}) = -2\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi};$$

but from (16)

$$\Gamma(-\tfrac{1}{2}) = \int_0^\infty \frac{e^{-x}dx}{x^{\frac{3}{2}}} = 2\int_0^\infty \frac{e^{-x}\, dx}{x^2},$$

by putting x^2 for x. Now, by Art. 122, the latter integral has an infinite element corresponding to $x = 0$, and, therefore, has an infinite value.

In fact we can take (16) as the definition of $\Gamma(n)$ for all positive values of n. The negative values must then be derived by means of (17).

It may be observed that $\Gamma(1) = 1$, and therefore $\Gamma(0) = \infty$. Hence $\Gamma(-p) = \infty$, where p is any integer.

127. To show that

$$\int_0^1 x^{n-1}(1-x)^{m-1}dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \tag{58}$$

We have, in effect, in Art. 106, already demonstrated this result for the case in which one of the quantities m, n is an integer.

From (16) we have, putting ax for x,

$$\Gamma(n) = \int_0^\infty a^n e^{-ax} x^{n-1}\,dx.$$

Multiplying both sides by $a^{m-1}e^{-a}\,da$, we get

$$\Gamma(n)a^{m-1}e^{-a}\,da = \int_0^\infty \{a^{m+n-1}e^{-a(1+x)}\,da\}\,x^{n-1}dx.$$

Hence, integrating with regard to a between ∞ and 0, we obtain

$$\Gamma(n)\int_0^\infty a^{m-1}e^{-a}\,da = \Gamma(n)\,\Gamma(m) = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\,x^{n-1}dx,$$

since $\displaystyle\int_0^\infty a^{m+n-1}e^{-a(1+x)}da = \Gamma(m+n)\,/\,(1+x)^{m+n}.$

But, putting $z/(1-z)$ for x, we find

$$\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^1 z^{n-1}(1-z)^{m-1}dz.$$

We thus have

$$\Gamma(n)\,\Gamma(m)=\Gamma(m+n)\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}}=\Gamma(m+n)\int_0^1 x^{n-1}(1-x)^{m-1}\,dx,$$

which gives the result stated above.

Let $m = 1 - n$, then

$$\int_0^\infty \frac{x^{n-1}\,dx}{1+x} = \Gamma(n)\,\Gamma(1-n).$$

Hence, from (44) we have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}. \qquad (59)$$

Putting $n = 1/2$, we get

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

which agrees with (24).

128. By means of (58) we can express many definite integrals in terms of Gamma functions.

Thus, putting $x = \sin^2\theta$, we have

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx = 2\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2n-1}(\cos\theta)^{2m-1}d\theta;$$

therefore $$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2n-1}(\cos\theta)^{2m-1}d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}. \qquad (60)$$

If we take $m = 1/2$, we get

$$\int_0^{\frac{1}{2}\pi} (\sin\theta)^{2n-1}\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}. \qquad (61)$$

We can hence derive a property of the Gamma function; for putting $m = n$ in (60), we have

$$\int_0^{\frac{1}{2}\pi} (\sin\theta \cos\theta)^{2n-1} d\theta = \frac{\Gamma^2(n)}{2\Gamma(2n)};$$

but $$\int_0^{\frac{1}{2}\pi} (\sin\theta \cos\theta)^{2n-1} d\theta = \frac{1}{2^{2n-1}} \int_0^{\frac{1}{2}\pi} (\sin 2\theta)^{2n-1} d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\frac{1}{2}\pi} (\sin\phi)^{2n-1} d\phi = \frac{1}{2^{2n-1}} \frac{\sqrt{\pi}}{2} \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})},$$

putting $$2\theta = \phi.$$

We thus find

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n), \tag{62}$$

which is a particular case of a more general theorem we shall prove further on.

129. We give here another proof of the results (41) already arrived at in Art. 117. Multiplying (19) by $\cos\beta a\,da$, we have

$$\int_0^\infty \{e^{-ax}\cos\beta a\,da\} x^{n-1} dx = \frac{\Gamma(n)}{a^n}\cos\beta a\,da.$$

Integrating then, with regard to a between ∞ and 0, and observing that, from Ex. 12, Art. 103,

$$\int_0^\infty e^{-ax}\cos\beta a\,da = \frac{x}{\beta^2 + x^2},$$

we get

$$\int_0^\infty \frac{x^n\,dx}{\beta^2 + x^2} = \Gamma(n)\int_0^\infty \frac{\cos\beta a\,da}{a^n}.$$

But from (45) we have, putting x/β for x,

$$\int_0^\infty \frac{x^n dx}{\beta^2 + x^2} = \frac{\pi\beta^{n-1}}{2\cos\frac{n\pi}{2}}.$$

Hence, putting x for a, we have

$$\int_0^\infty \frac{\cos\beta x\, dx}{x^n} = \frac{\beta^{n-1}}{\Gamma(n)} \frac{\pi}{2\cos\frac{n\pi}{2}}.$$

Similarly we find

$$\int_0^\infty \frac{\sin\beta x\, dx}{x^n} = \frac{\beta^{n-1}}{\Gamma(n)} \frac{\pi}{2\sin\frac{n\pi}{2}}.$$

Taking (59) into consideration, we see that these results agree with (41).

130. To show that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{-\frac{1}{2}}, \qquad (63)$$

where n is any integer.

First, let n be odd; then if we substitute $1/n$, $2/n$, &c., as far as $(n-1)/2n$, successively in (59), and multiply all the results together, we get

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{\pi^{\frac{n-1}{2}}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{2n}}.$$

Now we have

$$\frac{1-x^n}{1-x} = \left(1 - 2x\cos\frac{2\pi}{n} + x^2\right)\left(1 - 2x\cos\frac{4\pi}{n} + x^2\right)\ldots$$
$$\left(1 - 2x\cos\frac{(n-1)\pi}{n} + x^2\right).$$

Hence, putting $x=1$, and for $(1-x^n)/(1-x)$, its true value n, we get

$$n = 4\sin^2\frac{\pi}{n} \cdot 4\sin^2\frac{2\pi}{n} \ldots 4\sin^2\frac{(n-1)\pi}{2n};$$

therefore, extracting the square root of both sides,

$$n^{\frac{1}{2}} = 2^{\frac{n-1}{2}} \sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \ldots \sin\frac{(n-1)\pi}{2n},$$

from which (63) follows immediately.

When n is even, we substitute $1/n$, $2/n$, &c., as far as $(n-2)/2n$ in (59), and in a similar manner we obtain the same result, the equation $\Gamma(1/2) = \sqrt{\pi}$ being multiplied in with the $(n-2)/2$ equations just mentioned.

131. Several properties of the Gamma function can be obtained most easily by means of another function $\phi(x)$, which is defined by the equation

$$\phi(x) = \frac{d}{dx}\log\Gamma(1+x). \tag{64}$$

Differentiating the equation

$$\Gamma(x+2) = (x+1)\,\Gamma(x+1),$$

we get

$$\phi(x+1) = \phi(x) + \frac{1}{x+1}. \tag{65}$$

Hence, $\phi(x) = \phi(x+1) - \dfrac{1}{x+1} = \phi(x+2) - \dfrac{1}{x+1} - \dfrac{1}{x+2}$

$$= \phi(x+n) - \left\{\frac{1}{x+1} + \frac{1}{x+2} + \ldots + \frac{1}{x+n}\right\}. \tag{66}$$

Taking, then, $x = 0$, we have, when n is an integer,

$$\phi(n) = -C + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \tag{67}$$

where $-C = \phi(0)$.

To find an expression for C, we have

$$\frac{d}{dx}\Gamma(1+x) = \phi(x)\,\Gamma(1+x) = \frac{d}{dx}\int_0^\infty z^x e^{-z}\,dz = \int_0^\infty z^x \log z\, e^{-z} dz\,;$$

whence, putting $x = 0$, we have

$$C^* = -\phi(0) = \int_0^\infty e^{-z} \log\left(\frac{1}{z}\right) dz. \tag{68}$$

Now from (67) we obviously have

$$\phi(n) = -C + \int_0^1 (1 + z + z^2 + \ldots + z^{n-1})\,dz = -C + \int_0^1 \frac{(1-z^n)\,dz}{1-z}, \tag{69}$$

which thus, we see, defines $\phi(n)$ for integer values of n. But this also gives $\phi(x)$, when x is not an integer; for we have

$$\begin{aligned} \phi(x+1) - \phi(x) &= \int_0^1 \frac{(1-z^{x+1})\,dz}{1-z} - \int_0^1 \frac{(1-z^x)\,dz}{1-z} \\ &= \int_0^1 \frac{(z^x - z^{x+1})\,dz}{(1-z)} = \int_0^1 z^x\,dz = \frac{1}{x+1}, \end{aligned}$$

which agrees with the functional equation (65).

---

* The value of C to ten decimal places is 0·5772156649.

Hence we have, in general, expanding $1/(1-z)$,

$$\phi(x) = -C + \int_0^1 (1-z^x)(1+z+z^2+\&c.)dz$$

$$= -C + \frac{x}{x+1} + \frac{1}{2}\frac{x}{x+2} + \frac{1}{3}\frac{x}{x+3} + \&c. \qquad (70)$$

Putting now for $\phi(x)$ its value from (64), and integrating from $x = 0$, we obtain

$$\log \Gamma(1+x) = -Cx + x - \log(1+x) + \tfrac{1}{2}x - \log(1+\tfrac{1}{2}x)$$
$$+ \tfrac{1}{3}x - \log(1+\tfrac{1}{3}x) + \&c.;$$

whence, raising e to the power of both sides,

$$\Gamma(1+x) = e^{-Cx}\frac{e^x}{(1+x)}\;\frac{e^{\frac{1}{2}x}}{(1+x/2)}\;\frac{e^{\frac{1}{3}x}}{(1+x/3)}\;\&c. \qquad (71)$$

Changing the sign of x in (71), and multiplying the results together, we get

$$\Gamma(1+x)\,\Gamma(1-x) = \frac{1}{(1-x^2)}\;\frac{1}{(1-x^2/2^2)}\;\frac{1}{(1-x^2/3^2)}\;\&c. = \frac{\pi x}{\sin \pi x},$$

as we have already shown.

132. From (70) we have, by differentiation,

$$\frac{d\phi(x)}{dx} = \frac{d^2}{dx^2}\log\Gamma(1+x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c.$$
$$= \psi(x+1), \text{ say.}$$

Then $$\psi(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \&c.,$$

and $$n^2\psi(nx) = \frac{1}{x^2} + \frac{1}{(x+1/n)^2} + \frac{1}{(x+2/n)^2} + \ldots + \frac{1}{(x+(n-1)/n)^2}$$

$$+ \frac{1}{(x+1)^2} + \frac{1}{(x+1/n+1)^2} + \frac{1}{(x+2/n+1)^2} + \text{\&c.}$$

$$+ \frac{1}{(x+2)^2} + \frac{1}{(x+1/n+2)^2} + \text{\&c.}$$

$$+ \text{\&c.}$$

$$= \psi(x) + \psi(x+1/n) + \ldots + \psi\{x+(n-1)/n\},$$

by summing each of the columns. Hence, substituting for $\psi(x)$ its value in terms of $\Gamma(x)$, integrating twice, and raising e to the power of both sides, we get

$$Ae^{ax}\Gamma(nx) = \Gamma(x)\,\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right),$$

where A and a are constants. To determine A, let $x = 0$; then observing that

$$\Gamma(x)/\Gamma(nx) = n\Gamma(x+1)/\Gamma(nx+1) = n,$$

when $x = 0$, we have

$$A = n\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}},$$

from (63). Also putting $x = 1/n$, we find

$$Ae^{\frac{a}{n}} = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = A/n;$$

therefore $e^a = n^{-n}$. Hence we have, finally,

$$(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\Gamma(nx) = \Gamma(x)\,\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right) \tag{72}$$

We could give a more satisfactory demonstration of this result, but its length would be inconsistent with the limits of the present treatise. It will be observed that the weak point in the proof given above lies in the identification of the definite integral in (69) with the function $\phi(x)$.

133. To find an approximate value of $\Gamma(1+x)$, when x is very large. Putting $x=1$ in (71), we have

$$1 = e^{-C}\frac{e}{2}\,\frac{2e^{\frac{1}{2}}}{3}\,\frac{3e^{\frac{1}{3}}}{4}\,\frac{4e^{\frac{1}{4}}}{5}\cdots,$$

from which we get

$$\log(n+1) = -C + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{n} = \phi(n),$$

when n is infinite; therefore $\phi(n) = \log(n+1)$, approximately, when n is very large. Hence, since

$$\log(n+1) = \log n + \log\left(1+\frac{1}{n}\right) = \log n + \frac{1}{n} - \frac{1}{2n^2} + \&\text{c.},$$

we may suppose $\phi(n)$ to be capable of expansion in the form

$$\log n + \frac{A_1}{n} + \frac{A_2}{n^2} + \&\text{c.},$$

for large values of n. But, from (65),

$$\phi(n+1) - \phi(n) = \frac{1}{n+1} = \frac{1}{n} - \frac{1}{n^2} + \&\text{c.}$$

$$= \log\left(1+\frac{1}{n}\right) + \frac{A_1}{n+1} - \frac{A_1}{n} + \&\text{c.}$$

$$= \frac{1}{n} - \frac{1}{2n^2} + \&\text{c.} + \frac{A_1}{n} - \frac{A_1}{n^2} + \&\text{c.} - \frac{A_1}{n} + \&\text{c.};$$

hence, from the coefficient of $1/n^2$ we get $A_1 = 1/2$, so that we may write

$$\phi(n) = \log n + \frac{1}{2n} + \frac{A_2}{n^2} + \&c.,$$

or
$$\phi(x) = \log x + \frac{1}{2x} + \frac{A_2}{x^2} + \&c.,$$

if it be allowed to assume that $\phi(x)$ retains the same form when x is not an integer.

Hence, from (64) we have

$$\frac{d}{dx} \log \Gamma(1+x) = \log x + \frac{1}{2x} + \frac{A_2}{x^2} + \&c.\ ;$$

therefore, integrating, we get

$$\log \Gamma(1+x) = \text{a constant} + x \log x - x + \frac{1}{2} \log x - \frac{A_2}{x} + \&c.\ ;$$

or, omitting $-A_2/x$ and the following terms,

$$\Gamma(1+x) = Ax^x e^{-x} \sqrt{x},$$

when x is very large. To determine A, substitute the approximate values in (62), namely,

$$\Gamma(x)\, \Gamma(x+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2x-1}}\, \Gamma(2x),$$

or, as it may be written,

$$\Gamma(x+1)\, \Gamma(x+\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2^{2x+1}}\, (2x+1)\, \Gamma(2x+1),$$

and we get

$$A = \frac{\sqrt{\pi}}{2^{2x+1}} \frac{(2x+1)(2x)^{2x} e^{-2x} \sqrt{(2x)}}{\{x^x e^{-x} \sqrt{x}\}\, \{(x+\frac{1}{2})^{x+1} e^{-(x+\frac{1}{2})}\}},$$

which, by putting for $(x+1/2)^{x+1}$ its approximate value $x^{x+1}e^{\frac{1}{2}}$, gives

$$A = \sqrt{(2\pi)},$$

so that we have, finally,

$$\Gamma(1+x) = x^x e^{-x} \sqrt{(2\pi x)}, \tag{73}$$

approximately, for large values of x.

134. To express $\Gamma(1/4)$ and $\Gamma(3/4)$ in terms of a complete elliptic integral.

Taking $n = 1/4$, and $m = 1/2$ in (58), and observing that $\Gamma(1/2) = \sqrt{\pi}$, we get

$$\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{1}{\sqrt{\pi}}\int_0^1 \frac{dx}{x^{\frac{3}{4}}(1-x)^{\frac{1}{2}}} = \frac{4}{\sqrt{\pi}}\int_0^1 \frac{dz}{(1-z^4)^{\frac{1}{2}}},$$

putting $x = z^4$.

Now, from Ex. 4, Art. 85, we have

$$\int_0^1 \frac{dz}{\sqrt{(1-z^4)}} = \frac{K}{\sqrt{2}},$$

where the modulus is equal to $1/\sqrt{2}$.

Hence $$\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{2K\sqrt{2}}{\sqrt{\pi}};$$

but, from (59), taking $n = 1/4$,

$$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}.$$

We thus get

$$\left.\begin{aligned} \Gamma(\tfrac{1}{4}) &= 2\pi^{\frac{1}{4}}K^{\frac{1}{2}}, \\ \Gamma(\tfrac{3}{4}) &= \frac{\pi^{\frac{3}{4}}}{(2K)^{\frac{1}{2}}} \end{aligned}\right\}. \tag{74}$$

Examples.

1. $$\int_b^a \frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}} = \frac{\pi}{2}(a+b).$$

2. $$\int_0^\pi \frac{d\theta}{(1+\sin a\cos\theta)^3} = \frac{\pi}{2}\frac{(2+\sin^2 a)}{\cos^5 a}.$$

3. $$\int_0^{\frac{1}{2}\pi} \sin^3\theta\cos^7\theta\,d\theta = \frac{1}{40}.$$

4. $$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{\frac{1}{3}}\sin^5\theta\,d\theta = \frac{32}{231}.$$

5. $$\int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\{\sin\theta\cos^3\theta\}}} = \frac{12}{5}.$$

6. $$\int_0^\infty \frac{x^4\,dx}{(1+x^2)^5} = \frac{3\pi}{256}.$$

7. $$\int_0^\infty \frac{x^5\,dx}{(1+x^2)^6} = \frac{1}{60}.$$

8. $$\int_{-\infty}^\infty \frac{(q-3p^2+6pz)\,dz}{(2z^2+q-3p^2)^2\sqrt{(z^2+2pz+q)}} = \frac{1}{q-3p^2}.$$

9. $$\int_0^{\frac{1}{2}\pi} \cos\theta\cos 2\theta\cos 3\theta\,d\theta = \frac{\pi}{8}.$$

10. $$\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta\,d\theta}{1-\sin^2 a\cos^2\theta} = \frac{a-\sin a\cos a}{\sin^3 a\cos a}.$$

11. $$\int_0^a \sqrt{(\tan^2 a-\tan^2\theta)}\,d\theta = \frac{\pi}{2}(\sec a - 1).$$

12. $$\int_0^{\frac{1}{2}\pi} \theta^3\sin\theta\,d\theta = \frac{3\pi^2}{4} - 6.$$

13. $$\int_0^\infty \frac{dx}{x^4+2x^2\cos 2a+1} = \frac{\pi}{4\cos a}.$$

14. $$\int_0^1 \log x\log(1-x)\,dx = 2 - \frac{\pi^2}{6}.$$

15. $$\int_0^{\frac{1}{4}\pi} \tan\theta \log\cot\theta \, d\theta = \frac{\pi^2}{48}.$$

16. $$\int_0^{\frac{1}{2}\pi} \cot\theta \log\sec\theta \, d\theta = \frac{\pi^2}{24}.$$

17. $$\int_0^{\pi} \theta \sin\theta \cos(a\cos\theta) \, d\theta = \frac{\pi \sin a}{a}.$$

18. $$\int_0^{\pi} \theta \tan\tfrac{1}{2}\theta \log\sec\theta \, d\theta = \frac{\pi^3}{6}.$$

19. $$\int_0^{\pi} \theta \tan\theta \log\cot\tfrac{1}{2}\theta \, d\theta = \frac{\pi^3}{8}.$$

20. $$\int_0^{\pi} \theta \tan\theta \log\sin\theta \, d\theta = \frac{\pi^2}{24}.$$

21. $$\int_0^{\pi} \log(1 + e^{a\cos\theta}) \cos\theta \, d\theta = \frac{\pi a}{4}.$$

22. $$\int_0^{\frac{\pi}{2}} \sin n\theta \frac{(\cos\theta)^{n+1}}{\sin\theta} \, d\theta = \frac{\pi}{2^{n+1}}.$$

23. $$\int_0^{\infty} e^{-ax}(\sin x)^n \, dx = \frac{1 \,.\, 2 \,.\, 3 \ldots n}{a(a^2+1)(a^2+2^2)\ldots(a^2+n^2)},$$

when n is even; but

$$= \frac{1 \,.\, 2 \,.\, 3 \ldots n}{(a^2+1)(a^2+2^2)\ldots(a^2+n^2)},$$

when n is odd.

24. $$\int_0^{\pi} \cos ax (\sin x)^n \, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^n} \frac{\Gamma(n+1)}{\Gamma\left(1+\frac{n+a}{2}\right)\Gamma\left(1+\frac{n-a}{2}\right)}.$$

25. $$\int_0^{\pi} \sin ax (\sin x)^n \, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^n} \frac{\Gamma(n+1)}{\Gamma\left(1+\frac{n+a}{2}\right)\Gamma\left(1+\frac{n-a}{2}\right)}.$$

This and the preceding integral can be found from (67) and (68), Art. 71.

26. $$\int_0^1 \frac{(x^{m-1} + x^{n-1})\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

27. $$\int_0^{\frac{\pi}{2}} \frac{(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{2a^{2m}b^{2n}\,\Gamma(m+n)}.$$

28. $$\int_0^{\pi} \cos n\theta\,(\sin\theta)^n\,d\theta = \frac{\pi}{2^n}\cos\frac{n\pi}{2}.$$

29. $$\int_0^{\pi} \sin n\theta\,(\sin\theta)^n\,d\theta = \frac{\pi}{2^n}\sin\frac{n\pi}{2}.$$

30. $$\int_0^1 \left(\log\frac{1}{x}\right)^n \frac{dx}{1-x} = 1\,.\,2\,.\,3\,\ldots\,n\,.\,s_{n+1},$$

where $$s_n = \frac{1}{1} + \frac{1}{2^n} + \frac{1}{3^n} + \&c.$$

31. $$\int_0^1 x^{m-nx}\,dx = \frac{1}{m+1} + \frac{n}{(m+2)^2} + \frac{n^2}{(m+3)^3} + \&c.$$

32. $$\int_0^{\frac{1}{2}\pi} \frac{\log\sec\theta\,d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{\pi}{2ab}\log\left(1 + \frac{b}{a}\right).$$

33. $$\int_0^{\frac{1}{2}\pi} \frac{\log\tan\theta\,d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{\pi}{2ab}\log\left(\frac{a}{b}\right).$$

34. $$\int_0^{\pi} \epsilon^{a\cos\theta}\sin(\theta + a\sin\theta)\,d\theta = \frac{\epsilon^a - \epsilon^{-a}}{a}.$$

35. $$\int_0^{\pi} \theta\,\epsilon^{a\cos\theta}\sin(\theta + a\sin\theta)\,d\theta = \frac{\pi}{a}(1 - \epsilon^{-a}).$$

36. $$\int_0^{\frac{1}{2}\pi} \frac{\theta\cot\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{\pi}{2a^2}\log\left(1 + \frac{a}{b}\right).$$

37. Given $$I = \int_0^{\infty} e^{-x}\cos(a\log x)\,dx,$$

$$I' = \int_0^{\infty} e^{-x}\sin(a\log x)\,dx,$$

show that $$I^2 + I'^2 = \frac{2\pi a}{e^{\pi a} - e^{-\pi a}}.$$

This result is obtained by putting ia for a in (59).

38. $$\int_0^{\frac{1}{2}\pi} \frac{\sin n\theta\, d\theta}{\sin\theta} = \log 3 \sin n\pi - \log\left(\frac{5}{3}\right) \sin 2n\pi + \log\left(\frac{7}{5}\right) \sin 3n\pi$$

$$- \log\left(\frac{9}{7}\right) \sin 4n\pi + \&c.$$

$$= 2n \cos\frac{n\pi}{2} \left\{\frac{1}{1-n^2} - \frac{1}{3^2-n^2} + \frac{1}{5^2-n^2} - \frac{1}{7^2-n^2} + \&c.\right\}.$$

39. $$\int_0^{\frac{1}{2}\pi} \frac{(\cos\theta)^n \cos n\theta\, d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{\pi}{2}\,\frac{a^{n-1}}{b(a+b)^n}.$$

40. $$\int_0^{\pi} \frac{(\sin\theta)^{2n}\, d\theta}{(1 + a^2 - 2a\cos\theta)^n} = \int_0^{\pi} (\sin\theta)^{2n} d\theta, \text{ if } a < 1;$$

but $$= a^{-2n} \int_0^{\pi} (\sin\theta)^{2n} d\theta, \text{ if } a > 1.$$

41. $$\int_0^{\pi} \frac{\phi(\sin\theta)\, d\theta}{1 + e^{a\cos\theta}} = \frac{1}{2}\int_0^{\pi} \phi(\sin\theta)\, d\theta.$$

42. $$\int_0^1 \left(\frac{\sin^{-1}x}{x}\right)^3 dx = \frac{\pi}{2}\left(3\log 2 - \frac{\pi^2}{8}\right).$$

43. $$\int_0^{\frac{1}{2}\pi} \frac{\theta^2\, d\theta}{\sin^2\theta} = \pi \log 2.$$

44. $$\int_0^{\frac{1}{2}\pi} (\cos\theta)^{n-1} \frac{\sin n\theta}{\sin\theta}\, d\theta = \frac{\pi}{2}.$$

45. If $\sec\theta = 1 + A_2\theta^2 + A_4\theta^4 + \&c.$,

$$\int_0^{\infty} \frac{z^{2n}\, dz}{e^{\pi z} + e^{-\pi z}} = \frac{\underline{|2n}\,.\,A_{2n}}{2^{2n+2}}.$$

46. If $\tan\theta = \theta + A_3\theta^3 + A_5\theta^5 + \&c.$,

$$\int_0^{\infty} \frac{z^{2n-1}\, dz}{e^{\pi z} - e^{-\pi z}} = \frac{\underline{|2n-1}\,.\,A_{2n-1}}{2^{2n+1}}.$$

47. $$\int_0^\infty \frac{x^m\,dx}{x^2+2x\cos a+1} = \int_0^1 \frac{(x^m+x^{-m})\,dx}{x^2+2x\cos a+1} = \frac{\pi\sin ma}{\sin m\pi\sin a}.$$

This result may be deduced from

$$\int_0^\infty \frac{x^{m-1}dx}{1+x} = \frac{\pi}{\sin m\pi},$$

by putting xe^{ia} for x. It may also be obtained by putting x^n for x, and applying the method of decomposition into partial fractions, as in Art. 118.

48. $$\int_0^\infty \left(\frac{\sin rx}{x}\right)^n dx = \frac{\pi}{2^n}\frac{r^{n-1}}{\lfloor n-1}\left\{n^{n-1} - n(n-2)^{n-1} + \frac{n\,.\,n-1}{1\,.\,2}(n-4)^{n-1} - \&c.\right\}.$$

49. Show that

$$\int_0^{\frac{1}{2}\pi} \log\cot\theta\,d\theta = \int_0^1 \frac{dx}{x}\tan^{-1}x = \frac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{\theta\,d\theta}{\sin\theta}.$$

50. $$\int_0^{\frac{1}{2}\pi} (\log\cot\theta)^m d\theta = \lfloor m\, C_{m+1}, \quad \text{where} \quad C_m = \frac{1}{1^m} - \frac{1}{3^m} + \frac{1}{5^m} - \&c.$$

51. If $$S = 1 + \frac{nl}{m} + \frac{1}{1\,.\,2}\frac{n(n+1)\,l(l+1)}{m(m+1)} + \frac{1}{1\,.\,2\,.\,3}\frac{n(n+1)(n+2)\,l(l+1)(l+2)}{m(m+1)(m+2)} + \&c.,$$

show that $$S = \frac{\Gamma(m-l-n)\,\Gamma(m)}{\Gamma(m-n)\,\Gamma(m-l)}.$$

52. $$\frac{\Gamma(2n+1)}{\{\Gamma(n+1)\}^2} = 1 + n^2 + \left(\frac{n\,.\,n-1}{1\,.\,2}\right)^2 + \left(\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}\right)^2 + \&c.$$

53. $$\frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{\{\Gamma(1+\frac{1}{2}n)\}^2} = 1 - n^2 + \left(\frac{n\,.\,n-1}{1\,.\,2}\right)^2 - \left(\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}\right)^2 + \&c.$$

54. Let $$S = \frac{\sin n\theta}{n^2} - \frac{n\sin(n+2)\theta}{(n+2)^2} + \frac{n\,.\,n+1}{1\,.\,2}\frac{\sin(n+4)\theta}{(n+4)^2} - \frac{n\,.\,n+1\,.\,n+2}{1\,.\,2\,.\,3}\frac{\sin(n+6)\theta}{(n+6)^2} + \&c.;$$

then show that

$$S = \frac{\pi\theta}{2^{2n}}\frac{\Gamma(n)}{\{\Gamma(\frac{1}{2}+\frac{1}{2}n)\}^2}.$$

55. $$\frac{\Gamma(2n)}{\{\Gamma(n+1)\}^2} = 1 + \frac{1}{2}(n-1)^2 + \frac{1}{3}\left(\frac{n-1\,.\,n-2}{1\,.\,2}\right)^2 + \frac{1}{4}\left(\frac{n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3}\right)^2 + \text{\&c.}$$

56. Show, by putting $n = 1/3$, and $m = 1/2$ in (58), and taking account of (59), that

$$\Gamma(1/3),\quad \Gamma(2/3),\quad \Gamma(1/6),\quad \text{and}\quad \Gamma(5/6),$$

can be expressed in terms of the definite integral

$$\int_0^1 \frac{dx}{\sqrt{(1-x^3)}}.$$

57. Given
$$f(x) = a + bx + cx^2 + \text{\&c.},$$
$$\phi(x) = a' + b'x + c'x^2 + \text{\&c.},$$
show that
$$\int_0^\pi \{f(e^{i\theta})\,\phi(e^{-i\theta}) + f(e^{-i\theta})\,\phi(e^{i\theta})\}\,d\theta = 2\pi(aa' + bb' + cc' + \text{\&c.}).$$

58. $$\int_0^1 \left\{\frac{1-z^x}{1-z} - x\right\}\frac{dz}{\log z} = \log\Gamma(x+1).$$

59. $$\int_0^\infty \frac{\sin rx\,dx}{x(1+x^4)} = \frac{\pi}{2}\left\{1 - e^{-\frac{r}{\sqrt{2}}}\cos\frac{r}{\sqrt{2}}\right\}.$$

60. $$\int_0^\infty \frac{\cos rx\,dx}{1+x^4} = \frac{\pi}{2}\,e^{-\frac{r}{\sqrt{2}}}\sin\left(\frac{r}{\sqrt{2}} - \frac{\pi}{4}\right).$$

61. $$\int_0^\infty \frac{dx}{x^2+4\pi^2a^2}\log\frac{1}{1-e^{-x}} = \frac{1}{2a}\log\Gamma(1+a) + \frac{1}{2}(1-\log a) - \frac{1}{4a}\log(2\pi a).$$

62. $$\int_0^1 \frac{dx}{x}\log\frac{1}{1-\frac{1}{2}x} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2.$$

CHAPTER VII.

AREAS OF PLANE CURVES.

135. In the investigation of the values of portions of plane area bounded by geometrical figures, we commence by the use of the formula in rectangular Cartesian co-ordinates. If we consider an element PQ of a curve as the diagonal of the

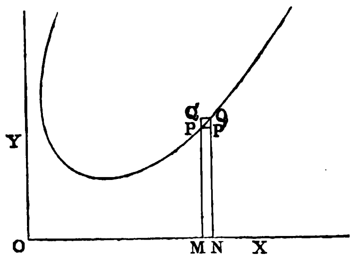

Fig. 4.

small rectangle formed by drawing parallels to the axes through P, Q, the area $PQMN$ is equal to $(y + \Delta y/2)\Delta x$, where $PM = y$, $OM = x$, $QP' = \Delta y$, $MN = \Delta x$. Hence, in the limit, neglecting $\Delta y\, \Delta x$, we have $dS = y dx$, and

$$S = \int_{x_2}^{x_1} y dx, \tag{1}$$

where S is the area between the curve, the two ordinates,

$$x - x_1 = 0, \quad x - x_2 = 0,$$

and the axis of x.

It is to be observed that in this result the whole portion of the curve between the limits must lie on the same side of the axis of x; for it is evident that the element of area for negative values of y is $-ydx$; so that if the curve crossed the axis between the limits, the definite integral would give the difference of the areas at opposite sides.

We have thus an immediate geometrical representation of any proposed definite integral

$$\int_b^a f(x)\,dx,$$

that is, the integral is equal to the portion of the area between the curve $y=f(x)$, the axis of x, and the ordinates $x=a$, $x=b$.

In a similar manner we find that the strip of area between an element of the curve, the axis of y, and two consecutive perpendiculars to that axis, is xdy. Hence, adding together the strips for both the axes, we have $ydx + xdy$, which is equal to $d(xy)$, or the differential of the rectangle formed by the co-ordinates, as, it is easy to see, it ought to be.

If the co-ordinates of a point on the curve are expressed as functions of a parameter θ, the formula (1) becomes

$$S = \int_{\theta_2}^{\theta_1} y\frac{dx}{d\theta}\,d\theta, \qquad (2)$$

where θ_1, θ_2 are the parameters of the extremities of the portion of the curve.

If the axes of co-ordinates are oblique, (1) must be replaced by

$$S = \sin\omega \int_{x_2}^{x_1} ydx, \qquad (3)$$

where ω is the angle between the axes.

136. If it be required to find the whole area of the space bounded by a closed curve, such as that represented in the figure, we may proceed as follows:—

Suppose the ordinate PM to meet the curve again in Q;

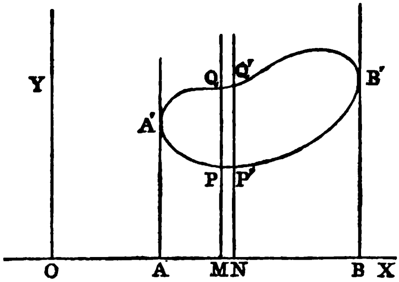

Fig. 5.

then if $PM = y_1$, $QM = y_2$, the area $PP'Q'Q$ is equal to $(y_2 - y_1)\,dx$. Now let a, b, be the abscissæ of the two extreme tangents AA', BB' of the curve drawn parallel to the axis of y; then, if S is the entire area,

$$S = \int_b^a (y_2 - y_1)\,dx. \tag{4}$$

This result, it is easy to see, still holds if the curve intersects the axis of x.

If the curve is symmetrical with regard to the axis of x, we evidently get

$$S = 2\int_b^a y\,dx. \tag{5}$$

137. As an example of the use of the formulæ in the preceding Articles, let us consider their application to the circle

$$x^2 + y^2 = a^2.$$

We have then

$$y = \sqrt{(a^2 - x^2)}, \text{ and } S = \int \sqrt{(a^2 - x^2)}dx,$$

taken between proper limits.

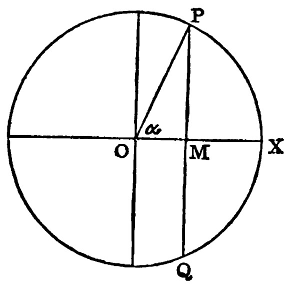

Fig. 6.

If then S is the area cut off by a chord PQ, we have

$$S = 2\int_{x'}^{a} \sqrt{(a^2 - x^2)}dx,$$

where $OM = x'$. Hence, putting $x = a\cos\theta$, we get

$$S = 2a^2 \int_0^{a} \sin^2\theta\, d\theta = a^2(a - \sin a \cos a),$$

where $$a = \angle POM.$$

Putting, now, $a = \pi$, we find that the whole area of the circle is πa^2.

Proceeding to the case of the ellipse, whose equation may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have $$ydx = \frac{b}{a}\sqrt{(a^2 - x^2)}dx,$$

which is equal to b/a multiplied by the corresponding element in the case of the circle $x^2 + y^2 = a^2$. Hence the area of any portion of the ellipse cut off by a perpendicular to the transverse axis is equal to b/a times the area cut off from the circle just mentioned. In the same way we see that a parallel to the transverse axis of the ellipse cuts off from the curve a portion of area which is a/b times the area cut off from the circle $x^2 + y^2 = b^2$.

We thus find that the entire area of the ellipse is πab.

We have already seen that the area cut off from the circle $x^2 + y^2 = a^2$ by a chord PQ is $a^2(\alpha - \sin\alpha\cos\alpha)$, where α is half the angle subtended by PQ at the centre. Hence we can deduce that the area cut off from the ellipse by any chord PQ is $ab(\alpha - \sin\alpha\cos\alpha)$, where α is half the difference of the eccentric angles of P, Q.

138. The method of deriving the area of the ellipse from that of the circle given in the preceding Article suggests a general principle, which is often useful in the determination of areas, viz.: the area of any portion of the curve $f(x/a, y/b) = 0$ is equal to ab multiplied by the corresponding area of the curve $f(x, y) = 0$. This follows from the fact, that the former curve is transformed into the latter by the transformation

$$x = ax', \quad y = by'.$$

From these equations we get $ydx = aby'dx'$, which gives the relation between the areas stated above.

139. Proceeding now to the case of the hyperbola, whose equation referred to its axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

we have
$$y = \frac{b}{a}\sqrt{(x^2 - a^2)},$$

and
$$S = \int y\,dx = \frac{b}{a}\int \sqrt{(x^2 - a^2)}\,dx.$$

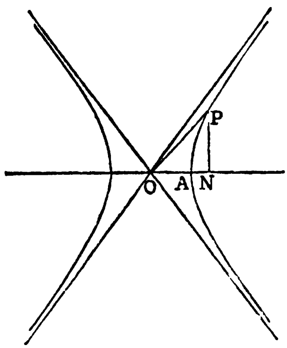

Fig. 7.

Hence, from (24), Art. 17,

$$S = \frac{bx}{2a}\sqrt{(x^2 - a^2)} - \frac{ab}{2}\log\left(\frac{x + \sqrt{(x^2 - a^2)}}{a}\right)$$

$$= \frac{xy}{2} - \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right),$$

where the area is supposed to be measured from the vertex A, that is, S is the area APN.

Hence, since the area OPN is equal to $xy/2$, we have

$$\text{area } POA = \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right).$$

It may be observed that we have thus a simple geometrical representation of a logarithm by means of the area

bounded by the arc of an hyperbola and two central radii vectores.

If the hyperbola is referred to its asymptotes, as axes of co-ordinates, its equation is $xy = k^2$; and from (3) we have then

$$S = k^2 \sin\omega \int \frac{dx}{x} = k^2 \sin\omega \log\left(\frac{x_1}{x_2}\right),$$

where ω is the angle between the asymptotes, and x_1, x_2 are the abscissæ of the bounding lines.

140. In the case of the parabola, if we refer the curve to its axis and the tangent at the vertex, we have $y^2 = px$, and the area APN is equal to

$$p^{\frac{1}{2}} \int x^{\frac{1}{2}} dx = 2p^{\frac{1}{2}} x^{\frac{3}{2}}/3 = 2xy/3.$$

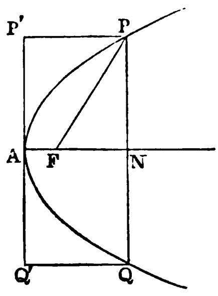

Fig. 8.

Hence we see that the area cut off from the curve by the line PQ, perpendicular to the axis, is two-thirds of the rectangle $PP'Q'Q$.

Again, since the equation of the parabola referred to a tangent and the corresponding diameter is $y^2 = p'x$, we can prove in the same way that the area cut off by any chord PQ

is two-thirds of the parallelogram formed by the chord, the parallel tangent, and the diameters drawn through P, Q.

141. As an example of the application of the formula (4), let us consider the curve $(y - mx^2)^2 = a^2 - x^2$. In this case,

$$y_1 - y_2 = 2\sqrt{(a^2 - x^2)}, \quad \text{and} \quad a^2 - x^2 = 0$$

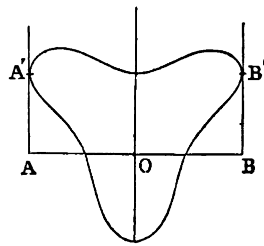

Fig. 9.

are the extreme tangents AA', BB' perpendicular to the axis of x.

If S then is the whole area, we have

$$S = 2\int_{-a}^{a} \sqrt{(a^2 - x^2)}\, dx = \pi a^2.$$

142. Again, as an example of (2), let us consider the curve $$x^3 + y^3 - 3axy = 0.$$

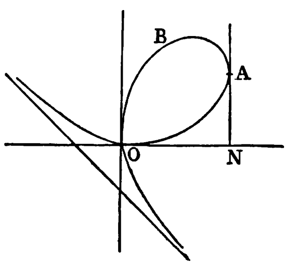

Fig. 10.

Putting $y = \theta x$, we get

$$x = \frac{3a\theta}{1+\theta^3}, \quad y = \frac{3a\theta^2}{1+\theta^3};$$

hence
$$dx = \frac{3a(1-2\theta^3)\,d\theta}{(1+\theta^3)^2},$$

and
$$S = \int y\,dx = 9a^2 \int \frac{\theta^2(1-2\theta^3)\,d\theta}{(1+\theta^3)^3} = C - \frac{9a^2}{2(1+\theta^3)^2} + \frac{6a^2}{1+\theta^3}.$$

Now for the origin O, which is a node of the curve, $\theta = 0$, and also $\theta = \infty$. Hence, taking S between these limits, we get $S = 3a^2/2$. But this, it is easy to see, is the difference of the areas $OBAN$, OAN, namely, the area of the loop OAB.

Examples.

1. If x, y are the co-ordinates of a point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that
$$\frac{x}{a} = \cos\left(\frac{S}{2ab}\right), \quad \frac{y}{b} = \sin\left(\frac{2S}{ab}\right),$$

where S is the sectorial area measured from the central radius and the transverse axis.

2. In the same case for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

show that

$$\frac{x}{a} = \frac{1}{2}\left(e^{\frac{S}{2ab}} + e^{-\frac{S}{2ab}}\right), \quad \frac{y}{b} = \frac{1}{2}\left(e^{\frac{S}{2ab}} - e^{-\frac{S}{2ab}}\right).$$

3. If a chord of a conic cut off a constant area from the curve, show that it touches a concentric, similar and similarly situated conic.

4. Show that the arc of the curve $a^{n-1}y = x^n$, where n is positive, measured from the origin to a point P, divides the rectangle formed by the axes and the perpendiculars from P on the axes into two parts, whose areas are in a constant ratio.

5. If the equation

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

represents an ellipse, show that its area is

$$\frac{\pi(af^2 + bg + ch^2 - 2fgh - abc)}{(ab - h^2)^{\frac{3}{2}}}.$$

6. Show that the whole area of the curve

$$x^2y^2 = (a - x)(x - b) \quad \text{is} \quad \pi(\sqrt{a} - \sqrt{b})^2/4.$$

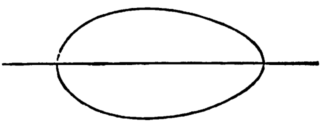

Fig. 11.

7. Show that the whole area of one of the ovals of the curve

$$x^2y^2 = (a^2 - x^2)(x^2 - b^2) \text{ is } \pi(a - b)^2/8.$$

8. Show that the whole area of the curve

$$y^2 = x^2(a - x)(x - b) \text{ is } \pi(a - b)^2(a + b)/8.$$

9. Show that the area between either branch of the curve

$$(xy - k^2)^2 = n^2x^2(a^2 - x^2),$$

and the axis of y is $n\pi a^2/2$.

10. Show that the whole area of either of the ovals of the curve

$$y^4 - 2y^2(a^2 + b^2 - 2x^2) + (a^2 - b^2)^2 = 0$$

is πb^2, where $b < a$.

11. Show that the area of either of the loops of the curve

$$y^4 - 2c^2 y^2 + a^2x^2 = 0 \text{ is } 8c^3/3a.$$

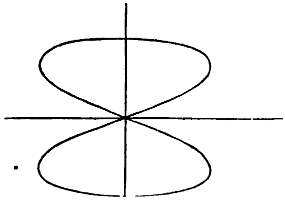

Fig. 12.

12. Show that the area between the curve

$$x(x^2 + y^2) = ay^2 + bx^2$$

and its asymptote $x - a = 0$ is $\pi(a - b)(3a + b)/4$, where $a > b$.

13. Show that the whole area between the curve $y(a^2 + x^2) = ma^3$, and the axis of x is $m\pi a^2$.

14. Show that the whole area between the curve $y^2(a^2 - x^2) = b^4$ and its asymptotes $x \pm a = 0$ is $2\pi b^2$.

15. Show that the area of one of the loops of the curve

$$y^4 - 2y^2(a^2 - b^2) + (a^2 + b^2 - 2x^2)^2 = 0$$

is
$$a^2 \cos^{-1}\left(\frac{\sqrt{(a^2 + b^2)}}{a\sqrt{2}}\right) - b^2 \log\left\{\frac{\sqrt{(a^2 + b^2)} + \sqrt{(a^2 - b^2)}}{b\sqrt{2}}\right\}.$$

16. Show that the whole area of the curve

$$(xy + a + bx^2)^2 = x^2(c^2 - x^2) \text{ is } \pi c^2.$$

17. Show that the whole area of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1 \text{ is } 3\pi ab/8.$$

This result may be most easily obtained by taking

$$x = a \sin^3\theta, \quad y = b \cos^3\theta,$$

and making use of (2).

18. Show that the whole area in the positive compartment between the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} = 1,$$

and the axes of co-ordinates is $ab/20$.

19. If S is the area between the curve $y^2 = a + bx^n$, the axes of co-ordinates, and the ordinate at a point xy, show that

$$S = \frac{2xy}{n+2} + \frac{na}{n+2}\int_0^x \frac{dx}{\sqrt{(a+bx^n)}}.$$

20. Show that the area between the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

the axes of co-ordinates and an ordinate is $c\sqrt{(y^2 - c^2)}$.

21. Show that the entire area of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta)$$

between the curve and its base is equal to $3\pi a^2$.

22. Show that the area between an arc PQ of the logarithmic curve $y = e^{cx}$ and the axis of x is proportional to the projection of PQ on the axis of y.

143. We now proceed to the determination of areas by means of polar co-ordinates.

Let APB be a curve referred to polar co-ordinates, and

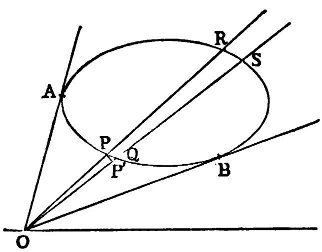

Fig. 13.

let OP, OQ be two consecutive radii vectores; then the area

OPQ is equal to the sum of the areas OPP', $PP'Q$, where P' is a point such that OP' is equal to OP. Now the latter area vanishes in the limit compared with OPP', which is evidently equal to $r^2 d\theta / 2$, where r, θ are the polar co-ordinates of P. Hence, if S is the area between the curve and two radii vectores, $\theta = a$, $\theta = \beta$, we have

$$S = \frac{1}{2} \int_\beta^a r^2 d\theta. \qquad (6)$$

This expression may also be found thus:—Double the area OPQ is, by analytic geometry, $x(y + dy) - y(x + dx)$, where x, y are the co-ordinates of P, and $x + dx$, $y + dy$ those of Q. Hence

$$2dS = xdy - ydx = r^2 d\theta,$$

by putting $\qquad x = r \cos\theta, \quad y = r \sin\theta.$

144. In finding the whole area of a closed curve by (6), we must consider separately the two cases in which O is outside or inside the curve. If, as in Fig. 13, O is outside the curve, we produce the radii vectores OP, OQ, to meet the curve again in R, S, respectively. Then the area

$$PQRS = ORS - OPQ = \tfrac{1}{2}(r_2^2 - r_1^2) d\theta, \qquad (7)$$

where $\qquad OP = r_1, \quad OR = r_2;$

and by integrating this expression between the limits determined by the two extreme tangents OA, OB, which can be drawn through O to the curve, we find the whole area.

If the origin lie inside the curve, we have evidently

$$2S = \int_0^{2\pi} r_1^2 d\theta = \int_0^{2\pi} r_2^2 d\theta,$$

and $$2S = \int_0^\pi (r_1^2 + r_2^2)d\theta. \tag{8}$$

The results in these two cases are most easily exemplified by applying them to the circle

$$r^2 - 2\delta r \cos\theta + \delta^2 - a^2 = 0.$$

We have then

$$r_1 + r_2 = 2\delta \cos\theta, \quad r_1 r_2 = \delta^2 - a^2,$$

and $$r_2 - r_1 = 2\sqrt{(a^2 - \delta^2 \sin^2\theta)};$$

so that, if the origin is outside the curve, we get

$$S = \frac{1}{2}\int_{-\alpha}^{\alpha} (r_2^2 - r_1^2)d\theta = 2\delta \int_{-\alpha}^{\alpha} \cos\theta \sqrt{(a^2 - \delta^2 \sin^2\theta)}d\theta,$$

where $$a = \delta \sin\alpha.$$

Hence, putting $\delta \sin\theta = a \sin\phi$, we find

$$S = 2a^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\phi \, d\phi = \pi a^2.$$

If the origin is inside the curve, δ is $< a$, and

$$r_1^2 + r_2^2 = 2a^2 + 2\delta^2 \cos 2\theta.$$

Hence $$S = \frac{1}{2}\int_0^\pi (r_1^2 + r_2^2)d\theta = \int_0^\pi (a^2 + \delta^2 \cos 2\theta)d\theta = \pi a^2.$$

145. As a further example, let us consider the pedal of a hyperbola with regard to its centre, namely, the curve whose equation is

$$r^2 = a^2 \cos^2\theta - b^2 \sin^2\theta.$$

In this case if S is the area of one of the loops, we evidently have

$$S = \frac{1}{2}\int_{-\alpha}^{\alpha} (a^2\cos^2\theta - b^2\sin^2\theta)d\theta,$$

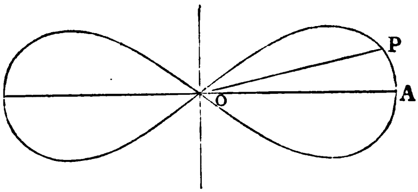

Fig. 14.

where α is the angle which either tangent at the node O makes with OA, that is, $\tan\alpha = a/b$. Hence

$$S = \frac{1}{2}ab + \frac{1}{2}(a^2 - b^2)\tan^{-1}\left(\frac{a}{b}\right).$$

Examples.

1. Show that the whole area of the curve

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta \text{ is } \pi(a^2+b^2)/2.$$

2. Show that the area between the Lemniscate $r^2 = 2c^2\cos 2\theta$ and the radii vectores $\theta = \alpha$, $\theta = \beta$, is

$$c^2\sin(\alpha-\beta)\cos(\alpha+\beta).$$

3. Show that the whole area bounded by the curve

$$(x^2+y^2-k^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) - (x^2+y^2) = 0 \text{ is } \pi(k^2+ab).$$

4. If $b < a$, show that the whole area of the curve

$$r = a + b\cos\theta \text{ is } \pi(a^2+b^2/2);$$

and if $b > a$, show that the area of the inner loop is

$$(a^2 + b^2/2)\alpha - 3a^2 \sin\alpha \cos\alpha/2,$$

and that the area of the space between the loops is

$$(2a^2 + b^2)\left(\frac{\pi}{2} - \alpha\right) + 3a^2 \sin\alpha \cos\alpha,$$

where $$b = a \cos\alpha.$$

5. Show that the area cut off in the positive compartment from the cubic $x(x^2 + y^2) - a^2 y = 0$ by any radius vector $y - x\tan\theta = 0$ is

$$\tfrac{1}{2}a^2 \log\sec\theta.$$

6. Show that the sectorial area of the curve

$$x^4 + y^4 + 2x^2y^2 \cos 2\alpha = a^4,$$

measured from the axis of x is

$$\tfrac{1}{4}a^2 F_k(\phi),$$

where $$k = \sin\alpha, \quad \tan\phi = 2xy/(x^2 - y^2).$$

7. Show that the whole area of the closed curve

$$\Sigma_0^n \frac{1}{\dfrac{x^2}{a_i^2} + \dfrac{y^2}{b_i^2}} = 1 \text{ is } \pi\Sigma_0^n a_i b_i.$$

8. Show that the area of either of the ovals of the curve

$$(x^2 + y^2)^2(a^2x^2 + b^2y^2) - k^6x^2 = 0 \text{ is } \frac{k^3}{\sqrt{(a^2 - b^2)}}\cos^{-1}\left(\frac{b}{a}\right),$$

where $a > b$.

9. Show that the whole area of the curve

$$(x^2 + y^2)^3 = (ax^2 + by^2)^2 \text{ is } \pi(3a^2 + 3b^2 + 2ab)/8,$$

where a, b have the same sign.

10. Show that the area of one of the loops of the curve

$$r^2 = a^2 \cos n\theta \text{ is } a^2/n.$$

11. Let P be a point on a branch of the cubic $r \cos 3\theta = a$, of which A is the summit; then, if O is the origin, and Q is the point of contact of one of the tangents drawn from P to the circle $r^2 - a^2 = 0$, show that the sectorial area POA is equal to a third of the area of the triangle POQ.

More generally for the curve $r \cos n\theta = a$, show that the sectorial area is equal to the n^{th} part of the area of the triangle.

12. Show that the whole area of the curve

$$(x^2 + y^2)^2 = ax^3 \quad \text{is} \quad 5\pi a^2/32.$$

13. Show that the area included between the curve and two radii vectores of the logarithmic spiral

$$r = ae^{c\theta} \quad \text{is} \quad (r^2 - r'^2)/4c.$$

14. In the hyperbolic spiral $r\theta = a$, show that the area bounded by the curve and two radii vectores is proportional to the difference of the lengths of these lines.

146. We mention here the formula for the sectorial area in terms of the radius vector and the perpendicular on the tangent, namely,

$$S = \frac{1}{2}\int \frac{pr\,dr}{\sqrt{(r^2 - p^2)}}. \tag{9}$$

This result is easily obtained; for $dS = pds/2$, where ds is the element PQ of the arc in Fig. 13, and $ds = dr \sec \phi$, $\sin \phi = p/r$, where ϕ is the angle PQP', namely, the angle which the radius vector makes with the curve.

If we put $p = r \sin \phi$ in the above, we get

$$S = \frac{1}{2}\int r \tan \phi \, dr. \tag{10}$$

These formulæ are of considerable use in cases in which the curve is such that p and r are connected by a simple relation.

For example, let us consider the involute of the circle. Let P be a point on the involute, then the tangent PT

to the circle is the normal to the curve; so that we have

$$p^2 = r^2 - a^2, \text{ where } OQ = p, \ OP = r, \ OT = a.$$

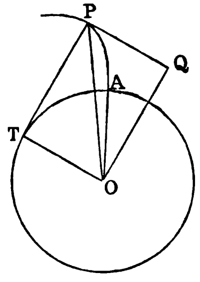

Fig. 15.

Hence, from (9), we get

$$S = \frac{1}{2a}\int r\sqrt{(r^2 - a^2)}\, dr = \frac{(r^2 - a^2)^{\frac{3}{2}}}{6a} = \frac{p^3}{6a},$$

if the area be measured from the line OA, where A is the point where the involute meets the circle.

Examples.

1. Show that the sectorial area of the curve $r = a + b \sin \phi$, measured from a tangent drawn from the origin, is

$$ab \sin^2 \tfrac{1}{2}\phi + \tfrac{1}{2} b^2 (\phi - \sin\phi \cos\phi).$$

2. Show that the sectorial area of the curve $r^2 = a^2 + b^2 \sin\phi$, measured from a tangent drawn from the origin, is

$$\tfrac{1}{2} b^2 \sin^2 \tfrac{1}{2}\phi.$$

3. To find the area of the epicycloid.

In this case, we have

$$r^2 = a^2 + \frac{4mp^2}{(m+1)^2},$$

so that we get

$$S = \frac{1}{2}\int \frac{(m+1)\sqrt{(r^2-a^2)}\,rdr}{\sqrt{\{(m+1)^2a^2-(m-1)^2r^2\}}};$$

which, if we put

$$r^2 = a^2\cos^2\theta + \frac{a^2(m+1)^2}{(m-1)^2}\sin^2\theta,$$

gives

$$S = \frac{2m(m+1)a^2}{(m-1)^3}\int \sin^2\theta\, d\theta$$

$$= \frac{m(m+1)a^2}{(m-1)^3}(\theta - \sin\theta\cos\theta),$$

if the area be measured from the fixed circle. Hence, taking $\theta = \pi$, we find that the area between the curve and the radii vectores to two consecutive cusps is

$$\frac{m(m+1)\pi a^2}{(m-1)^3}.$$

147. We now proceed to consider the area of the general cubic, and shall show that in all cases it can be expressed by means of no higher transcendents than elliptic integrals. Let us take the axis of y parallel to the real asymptote which the cubic must always have, then it is shown in treatises on curves or the Differential Calculus that the cubic can be written

$$y^2(ax+b) + y(a'x^2+b'x+c') + a''x^3+b''x^2+c''x+d'' = 0. \quad (11)$$

Furthermore, if the axis of y is the asymptote itself, $b = 0$. Solving now this equation for y, if y_1, y_2 are the roots, we have

$$y_1 - y_2 = \frac{\sqrt{X}}{ax},$$

where we have put

$$(a'x^2+b'x+c')^2 - 4ax(a''x^3+b''x^2+c''x+d'') = X.$$

Hence, from (4), we get

$$\int_{x_2}^{x_1} \frac{\sqrt{X}}{ax} dx, \qquad (12)$$

for the area between the curve and two parallels to the real asymptote; and this expression by Chapter V. can be expressed by elliptic integrals.

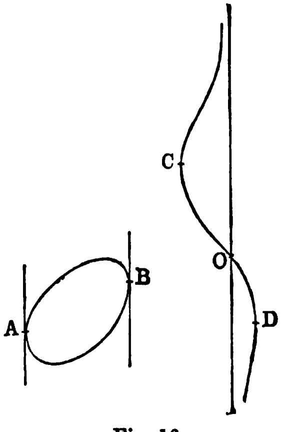

Fig. 16.

If the curve consist of an oval and an infinite serpentine branch, as in Fig. 16, the area of the oval will be found by integrating between the limits of the extreme tangents. These tangents are evidently determined by two roots α, β, say, of $X = 0$; so that if we put

$$X = (\alpha - x)(x - \beta)(lx^2 + 2mx + n),$$

we have

$$\frac{1}{a}\int_{\beta}^{\alpha} \sqrt{\{(\alpha - x)(x - \beta)(lx^2 + 2mx + n)\}}\, \frac{dx}{x} \qquad (13)$$

for the area of the oval.

148. With respect to the area between the curve and the asymptote, that on one side will be obtained by integrating between the limits γ, 0, and that on the other by integrating between δ, 0, where γ, δ are the values of x corresponding to the points C, D on the infinite branch, at which the tangents are parallel to the asymptote. Applying now the criterion of Art. 122, the integral (12) is found to be infinite in both these cases; so that, in general, the corresponding areas are also infinite.

If, however, the line of infinity is an inflexional tangent of the curve, that is, if $c' = 0$, in which case the points O, D remove themselves to infinity, and the infinite branch lies altogether on one side of the asymptote, we may put

$$X = m^2 x (a - x)(\beta - x)(\gamma - x),$$

and the whole area between the infinite branch and the asymptote is then

$$\frac{m}{a}\int_0^\gamma \sqrt{\left\{\frac{(a - x)(\beta - x)(\gamma - x)}{x}\right\}}\,dx, \qquad (14)$$

which, by Art. 122, has a finite value.

If the serpentine branch is replaced by three hyperbolic branches, we have still an expression such as (13) for the area of the oval. And there is no difficulty in finding the integrals which express the areas corresponding to each of the various forms which a cubic curve can assume. For these forms and their figures, we refer the reader to Salmon's *Treatise on the Higher Plane Curves*, Arts. 199–209.

149. It may be of some interest to notice here those cubic curves whose areas can be expressed by logarithmic or circular functions. First of all, this will be the case if the

curve have a node; for then X will have $(x-\delta)^2$ as a factor, where δ is the abscissa of the node, and therefore the radical

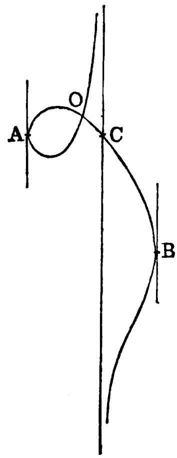

Fig. 17.

in the integral (13) will be only of the second degree in x. If the curve is as in Fig. 17, we may put

$$X = m^2 (a-x)(x-\beta)(x-\delta)^2,$$

and the area of the loop will evidently be

$$\frac{m}{a}\int_\delta^a \frac{(x-\delta)}{x}\sqrt{\{(a-x)(x-\beta)\}}\,dx; \qquad (15)$$

where a, β determine the points A, B at which the tangents are parallel to the asymptote.

Again, it is easy to see that (12) will depend upon lower integrals if X involve only even powers of the variable.

Putting $y + h$ for y in (11), we can determine h so that b' may vanish; and taking, then, $b'' = d'' = 0$, the curve may be written

$$axy^2 + y(a'x^2 + c') + x(a''x^2 + c'') = 0; \qquad (16)$$

and X being then

$$(a'x^2 + c')^2 - 4ax^2(a''x^2 + c''),$$

the expression for the area is of the form

$$\int \sqrt{(az^2 + \beta z + \gamma)}\,\frac{dz}{z}, \qquad (17)$$

where we have put $x^2 = z$.

The area in this case is, however, most easily obtained by using the formula for the elementary sectorial area, namely,

$$dS = (xdy - ydx)/2.$$

We have then, from (16),

$$S = \frac{1}{2}\int(xdy - ydx) = \frac{1}{2}\int\frac{(ydx - xdy)(c'y + c''x)}{x(ay^2 + a'xy + a''x^2)}, \qquad (18)$$

which gives the area at once as the integral of a rational expression.

It may be observed that (16) is called Chasles' central cubic. There are five kinds of these curves, according to the nature of the factors of the denominator of the fraction involved in (18), which have to be considered separately, when we seek the evaluation of the integral. (See, *loc. cit.*, Art. 197.)

EXAMPLES.

1. Show that the whole area of the oval of the curve $y^2 = x(1-x)(1-k^2x)$, where $k < 1$, is

$$\frac{4(k^4 - k^2 + 1)}{15k^2} E_k - \frac{2(1-k^2)(2-k^2)}{15k^2} F_k.$$

2. If all the cubics of the system

$$x(y + \alpha x + \beta)^2 + 2k(y + \alpha x + \beta) + ax^3 + bx^2 + cx + d = 0,$$

where α, β are variable, have oval figures, show that these ovals have the same area.

3. Show that the whole area of the loop of the cubic

$$x(x^2 + y^2) - (cx^2 - ay^2) = 0$$

is $$\tfrac{1}{2} a^2 \tan\theta\,(3 + \tan^2\theta) - \tfrac{1}{2} a^2\,\theta \sec^2\theta\,(3 - \tan^2\theta),$$

where $$c = a\tan^2\theta.$$

4. If two perpendiculars to the axis of x meet the cubic $xy^2 = a^3$, show that the area cut off from the curve is proportional to the difference of the reciprocals of the intercepted chords. Also show that the area cut off by the chord joining two points, y_1, y_2, lying on the same side of the axis of x, is

$$\frac{a^3(y_1 - y_2)^3}{2y_1^2y_2^2}.$$

5. Show that the area cut off from the cubic $a^2y = x^3$ by the chord joining two points, x_1, x_2, is

$$\frac{(x_1 - x_2)^3(x_1 + x_2)}{4a^2}.$$

6. Show that the sectorial area of the cubic

$$x(x^2 + 2bxy + cy^2) - k^2(bx + cy) = 0$$

described about the origin, and measured from the axis of x, is

$$\frac{1}{4} k^2 \log\left(\frac{x^2 + 2bxy + cy^2}{x^2}\right).$$

150. Proceeding now to curves of the fourth order, we remark that, as a particular case of Clebsch's theory of curves, the area of the general curve can be expressed by Abelian integrals, and of the curve with a node by hyper-elliptic integrals; also that the area of the curve with two nodes depends upon elliptic integrals. Some cases of the latter curve, or binodal quartic, as it is called, are, therefore, the most general curves of the fourth order, which we shall consider here.

Of the binodal quartics, the most interesting is the curve called the bicircular quartic. This curve may be generated in the following manner:—

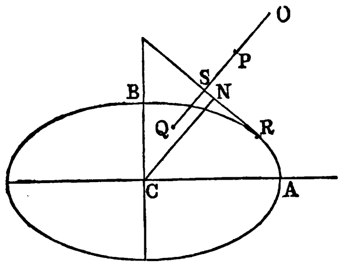

Fig. 18.

Given a conic, as in Fig. 18, and a fixed point O, let S be the foot of the perpendicular from O on the tangent at a point R of the conic, and let P, Q be two points taken on OS, so that $OQ = p + \sqrt{(p^2 - k^2)}$, $OP = p - \sqrt{(p^2 - k^2)}$, where $OS = p$, and k is a constant; then, as R moves round the conic, the locus of P, Q is a bicircular quartic.

By using the formula, then, of polar co-ordinates, we have at once the expressions for the areas swept out by OP, OQ about the point O, namely,

$$S = \tfrac{1}{2}\int\{p + \sqrt{(p^2 - k^2)}\}^2 d\omega, \quad S' = \tfrac{1}{2}\int\{p - \sqrt{(p^2 - k^2)}\}^2 d\omega,$$

where ω is the angle which OS makes with OA. Hence we get

$$S + S' = \int (2p^2 - k^2)\, d\omega, \tag{19}$$

$$S - S' = 2\int p\sqrt{(p^2 - k^2)}\, d\omega. \tag{20}$$

Now, if the curve be an ellipse, as in the figure, we may write its equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

and the tangent at the point $a\cos\phi$, $b\sin\phi$ is, then,

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0;$$

whence $$\tan\omega = a\tan\phi / b,$$

and $$d\omega = ab\,d\phi / (a^2\sin^2\phi + b^2\cos^2\phi);$$

also $$p = \frac{ba\cos\phi + a\beta\sin\phi - ab}{\sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}},$$

where α, β are the co-ordinates of O.

We thus get

$$S + S' = ab\int (2T^2 - k^2 b'^2)\frac{d\phi}{b'^4}, \tag{21}$$

$$S - S' = 2ab\int T\sqrt{(T^2 - k^2 b'^2)}\frac{d\phi}{b'^4}, \tag{22}$$

where

$$ba\cos\phi + a\beta\sin\phi - ab = T, \quad a^2\sin^2\phi + b^2\cos^2\phi = b'^2.$$

But putting $\tan\frac{1}{2}\phi = \mu$, we have

$$\cos\phi = \frac{1-\mu^2}{1+\mu^2}, \quad \sin\phi = \frac{2\mu}{1+\mu^2}, \quad d\phi = \frac{2d\mu}{1+\mu^2};$$

and substituting these values in (21), we see that the sum of the areas is expressed by the integral of a rational function of μ, and, therefore, depends upon logarithmic and circular functions. Again, by making the same substitutions in (22), the difference of the areas is expressed by an integral involving the square root of a polynomial in μ of the fourth degree. This difference, therefore, depends upon elliptic integrals.

We may observe that the integral which gives the sum of the areas is most conveniently expressed in terms of the angle ω. We have

$$p + CN = \alpha \cos \omega + \beta \sin \omega,$$

or
$$p = \alpha \cos \omega + \beta \sin \omega - \sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)},$$

since
$$CN = \sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)};$$

hence
$$\begin{aligned} S + S' = \int \{ & \alpha^2 + \beta^2 + a^2 + b^2 - k^2 + (\alpha^2 - \beta^2 + a^2 - b^2) \cos 2\omega \\ & + 2\alpha\beta \sin 2\omega - 4 (\alpha \cos \omega + \beta \sin \omega) \sqrt{(a^2 \cos^2 \omega} \\ & + b^2 \sin^2 \omega) \} \, d\omega. \end{aligned} \tag{23}$$

151. When the quartic has no finite double point, there are three fundamentally distinct figures possible, namely, two ovals, one of which is wholly inside the other, two ovals exterior to each other, and thirdly a single oval. In the first of these cases we find the sum of the areas of the two ovals by effecting the integration in (23) between the limits 2π and 0, for then the point O lies within the ellipse, and within the inner oval. We thus get

$$S + S' = 2\pi (\alpha^2 + \beta^2 + a^2 + b^2 - k^2). \tag{24}$$

In the second case we proceed to show that the difference of the areas of the ovals can be readily obtained. The gene-

rating conic is now a hyperbola, and the curve is as represented in Fig. 19. Drawing then two parallel tangents to

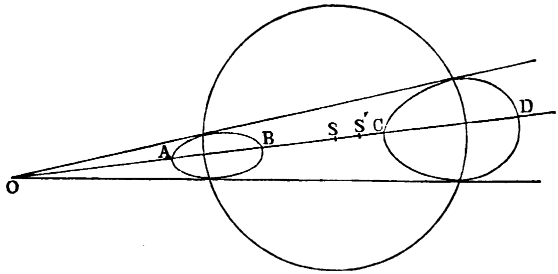

Fig. 19.

the hyperbola, we get the four points A, B, C, D, where a radius vector through O meets the curve. Hence, if

$$\alpha \cos \omega + \beta \sin \omega = q, \quad \sqrt{(a^2 \cos^2\omega - b^2 \sin^2\omega)} = \Delta,$$

where the sign of b^2 is changed, we get

$$OA + OD = 2OS' = 2(q + \Delta), \text{ and } OB + OC = 2OS = 2(q - \Delta);$$

therefore, since

$$OA \,.\, OD = k^2 = OB \,.\, OC,$$

we have

$$OD^2 - OC^2 - (OB^2 - OA^2) = 16q\Delta.$$

But
$$S = \tfrac{1}{2}\int (OD^2 - OC^2)\, d\omega,$$

$$S' = \tfrac{1}{2}\int (OB^2 - OA^2)\, d\omega,$$

where the limiting values of ω are those given by $\Delta = 0$, namely,

$$\tan \omega = \pm\, a/b = \tan \lambda, \text{ say.}$$

Hence

$$S - S' = 8\int_{-\lambda}^{\lambda} (a \cos\omega + \beta \sin\omega)\sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}\,d\omega$$

$$= 8a\int_{-\lambda}^{\lambda} \sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}\,d(\sin\omega)$$

$$-8\beta\int_{-\lambda}^{\lambda} \sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}\,d(\cos\omega);$$

but the latter of these integrals vanishes between the limits, and

$$\int_{-\lambda}^{\lambda} \sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}\,d(\sin\omega)$$

$$= 2\sqrt{(a^2+b^2)}\int_0^{\lambda} \sqrt{(\sin^2\lambda - \sin^2\omega)}\,d(\sin\omega)$$

$$= 2\sqrt{(a^2+b^2)}\sin^2\lambda\int_0^{\frac{1}{2}\pi}\cos^2\phi\,d\phi = \frac{\pi a^2}{2\sqrt{(a^2+b^2)}},$$

by putting $\sin\omega = \sin\lambda\sin\phi$. We thus get

$$S - S' = \frac{4\pi a^2 a}{\sqrt{(a^2+b^2)}}. \tag{25}$$

152. If in the preceding mode of generation of the quartic the constant k vanishes, the locus is evidently a curve similar to the pedal of the point O with regard to the conic, where by the name pedal we denote the locus of the feet of the perpendiculars drawn from a point to the tangents of a curve. In fact, then, in Fig. 18, P coincides with O, and

$$OQ = 2OS.$$

Now, since p, ω are the polar co-ordinates with regard to O of the point S on the pedal, we have for the area Σ of that curve

$$\Sigma = \tfrac{1}{2}\int p^2 d\omega;$$

so that the area of the pedal is equal to a fourth part of the area of the locus of Q.

If O lies within the ellipse, the inner oval shrinks into the point O, when k vanishes. Hence, by taking the fourth part of the area given in (24), we have for the area of the pedal

$$\Sigma = \frac{\pi}{2}(a^2 + b^2 + \alpha^2 + \beta^2).$$

Again, if O lies without the conic, the pedal has O for a crunode, and (24), it is easy to see, gives the sum of the areas of the two loops of which the pedal then consists. Also (25) gives the difference of the areas of these loops; so that we get for the area Σ of a loop the expression

$$\Sigma = \frac{\pi}{4}\left\{a^2 - b^2 + \alpha^2 + \beta^2 \pm \frac{2a^2\alpha}{\sqrt{(a^2+b^2)}}\right\}, \qquad (26)$$

where we have changed the sign of b^2.

153. We proceed now to consider the special case of the ellipse of Cassini, a curve which is defined as the locus of a point, the product of whose distances from two fixed points F, F' is constant. There are two forms of this curve to be considered separately, namely, if $FF' = 2c$, and the constant product equals m^2, when $m > c$, the curve consists of the single outer oval in the figure, and when $m < c$, it consists of the two conjugate inner ovals. Taking the origin at C,

the middle point of FF', the polar equation of the curve is

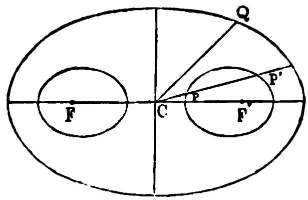

Fig. 20.

easily found to be

$$r^4 - 2c^2r^2\cos 2\theta + c^4 - m^4 = 0, \tag{27}$$

from which we get

$$r^2 = c^2\cos 2\theta \pm \sqrt{(m^4 - c^4\sin^2 2\theta)}.$$

When m is $> c$, we must always take the positive sign; and then if S is the whole area of the oval, we have

$$S = \tfrac{1}{2}\int_0^{2\pi} \{c^2\cos 2\theta + \sqrt{(m^4 - c^4\sin^2 2\theta)}\}\,d\theta = 2m^2E,$$

where the modulus of the elliptic integral is c^2/m^2.

When m is $< c$, if $CP' = r$, $CP = r'$,

$$r^2 - r'^2 = 2\sqrt{(m^4 - c^4\sin^2 2\theta)};$$

and if S is the whole area of either of the ovals, we have

$$S = \int_{-a}^{a} \sqrt{(m^4 - c^4\sin^2 2\theta)}\,d\theta,$$

where $\sin 2a = m^2/c^2$, that is, the angles $\pm a$ determine the tangents drawn from C to the oval.

Hence, putting $c^2 \sin 2\theta = m^2 \sin\phi$, we get

$$S = \int_0^{\frac{1}{2}\pi} \frac{m^4 \cos^2\phi d\phi}{\sqrt{(c^4 - m^4 \sin^2\phi)}} = c^2(E - k'^2 K),$$

where k, the modulus, is equal to m^2/c^2.

154. We now proceed to demonstrate a geometrical theorem connecting the area of the oval of Cassini with that of the Lemniscate; namely, if a variable concentric Lemniscate

$$r^2 = a \cos 2\theta + b \sin 2\theta$$

touch the Cassinian oval

$$r^4 - 2c^2 r^2 \cos 2\theta - k^4 = 0;$$

then it cuts off a constant area from the oval

$$r^4 - 2c^2 r^2 \cos 2\theta - k'^4 = 0,$$

where $k' < k$.

Let S be the area PAB cut off by the Lemniscate from

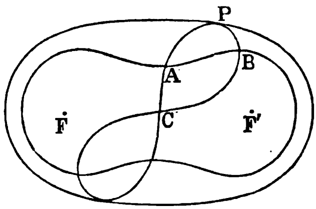

Fig. 21.

the curve $r^2 = \phi(\theta)$, then evidently

$$S = \frac{1}{2}\int_{\theta_2}^{\theta_1} \{a \cos 2\theta + b \sin 2\theta - \phi(\theta)\} d\theta,$$

where the limits θ_1, θ_2 correspond to the radii vectores CA, CB; that is, are determined by the equation

$$a \cos 2\theta + b \sin 2\theta - \phi(\theta) = 0.$$

Now if the Lemniscate ABC touch an envelope, a and b will be connected by a certain relation, so that we may consider both these quantities as functions of another variable t. If then the area remains constant while the Lemniscate varies, we must have

$$\frac{dS}{dt} = 0 = \frac{1}{2}\int_{\theta_2}^{\theta_1}\left(\frac{da}{dt}\cos 2\theta + \frac{db}{dt}\sin 2\theta\right)d\theta,$$

by Art. 121, as the quantities outside the sign of integration vanish by the equation which determines the limits. Hence, performing the integration and dividing by $\sin(\theta_1 - \theta_2)$, we get

$$\frac{da}{dt}\cos(\theta_1 + \theta_2) + \frac{db}{dt}\sin(\theta_1 + \theta_2) = 0.$$

But if we seek now the point of contact of the Lemniscate with its envelope, we have, by differentiating its equation with regard to t,

$$\frac{da}{dt}\cos 2\theta + \frac{db}{dt}\sin 2\theta = 0;$$

therefore, eliminating da/dt, db/dt from this and the preceding equation, we obtain $2\theta = \theta_1 + \theta_2$, or the radius vector CP drawn to the point of contact P of the Lemniscate with its envelope must bisect the angle between the radii vectores CA, CB. The theorem, then, which we have stated above will be demonstrated, if we show that the condition which we have just arrived at holds, when the envelope of the Lemniscate is a Cassinian oval, and the inner curve is another Cassinian, having the same foci, F, F'.

Now, eliminating r^2 between the equation of the Lemniscate,

$$r^2 = a\cos 2\theta + b\sin 2\theta$$

and that of the Cassinian

$$r^4 - 2c^2r^2\cos 2\theta - k'^4 = 0,$$

we get a result which may be written

$$(b^2 - k'^4)\tan^2 2\theta + 2b(a - c^2)\tan 2\theta + a^2 - 2c^2a - k'^4 = 0;$$

and this equation gives the values of $\tan 2\theta_1$, $\tan 2\theta_2$, so that we have

$$\tan 2\theta_1 + \tan 2\theta_2 = \frac{2b(c^2 - a)}{b^2 - k'^4},$$

$$\tan 2\theta_1 \tan 2\theta_2 = \frac{a^2 - 2c^2a - k'^4}{b^2 - k'^4},$$

from which we get

$$\tan 2(\theta_1 + \theta_2) = \frac{\tan 2\theta_1 + \tan 2\theta_2}{1 - \tan 2\theta_1 \tan 2\theta_2} = \frac{2b(c^2 - a)}{b^2 - a^2 + 2c^2a}.$$

We see thus that $\theta_1 + \theta_2$ is independent of k', and, therefore, equal to 2θ, where θ is the angle which determines the radius vector to the point where the Lemniscate is touched by a Cassinian oval of the system obtained by varying k'; and this is what was to be proved.

It is to be observed that this theorem will hold equally well if each of the Cassinians consists of a pair of ovals. In both cases, as we have seen, the area of the Cassinian depends upon elliptic integrals, and the area of the Lemniscate is algebraic, so that the results we have arrived at afford a simple geometrical illustration of the comparison theorems of these integrals.

155. If a quartic have two real nodes at infinity, its equation may be written in the form

$$y^2(ax^2 + bx + c) + y(a'x^2 + b'x + c') + a''x^2 + b''x + c'' = 0,$$

where the axes of co-ordinates are taken parallel to the direction of the nodes. In general the axes will be oblique, but there is evidently no loss of generality in supposing them to be rectangular in finding the value of the area, as the two expressions (1) and (3) are in the constant ratio $\sin \omega$.

Solving then this equation for y, if $y_1\, y_2$ are the roots, we get

$$y_1 - y_2 = \frac{\sqrt{X}}{ax^2 + bx + c},$$

where $\quad (a'x^2 + b'x + c')^2 - 4(ax^2 + bx + c)(a''x^2 + b''x + c'') = X,$

and we have then the expression

$$S = \int_{x_2}^{x_1} \frac{\sqrt{X}\,dx}{ax^2 + bx + c} \tag{28}$$

for the area intercepted between the curve and the lines

$$x - x_1 = 0, \quad x - x_2 = 0.$$

Examples.

1. If the quartic

$$(x^2 + y^2 - 2\beta y + k^2)^2 - 4(a^2x^2 - b^2y^2) = 0$$

consist of two ovals exterior to each other, show that their areas are equal.

2. The quartic

$$(x^2 + y^2 + k^2)^2 = 4(a^2x^2 + b^2y^2), \quad \text{where} \quad a > b > k,$$

consists of two concentric ovals; if S is the area cut off between the ovals, a radius vector $y = x \tan \omega$ and the axis of x, show that

$$S = \frac{b(a^2 - b^2)\sqrt{(a^2 - k^2)} \sin\phi \cos\phi \Delta(\phi)}{a^2 \sin^2\phi + b^2 \cos^2\phi} + b\sqrt{(a^2 - k^2)} E(\phi)$$

$$- \frac{a^2 b}{\sqrt{(a^2 - k^2)}} F(\phi) + \frac{a^2(a^2 + b^2 - k^2)}{b\sqrt{(a^2 - k^2)}} \Pi(n, \phi),$$

where $$\tan\phi = b\tan\omega/a, \quad n = (a^2-b^2)/b^2,$$

and λ, the modulus, is equal to

$$k\sqrt{(a^2-b^2)}/b\sqrt{(a^2-k^2)}.$$

3. The quartic

$$(x^2+y^2+a^2-b^2)^2 = 4(a^2x^2+b^2y^2), \quad \text{where} \quad 2b^2 > a^2,$$

consists of two concentric ovals; show that their areas are

$$b^2(3\pi+2E), \quad b^2(\pi-2E),$$

respectively, where the modulus is $(a^2-b^2)/b^2$.

4. If the Cartesian

$$r^2 - 2r(a+b\cos\theta) + k^2 = 0$$

consists of two ovals, one of which is wholly within the other, show that the sum of their areas is equal to $2\pi(2a^2+b^2-k^2)$.

5. If $k^4 > a^2b^2$, show that the whole area between the curve

$$(a^2-x^2)(y^2+b^2) = k^4 \text{ and the lines } a^2-x^2=0 \quad \text{is} \quad 4k^2E,$$

where the modulus is ab/k^2.

If $k^4 < a^2b^2$, show that the area between one branch of the curve and the adjacent asymptote is $2k^2\{E-(1-\lambda^2)K\}$, where the modulus $\lambda = k^2/ab$.

6. Show that the whole area of the loop of the curve

$$x^2y^2 + a^2x^2 + b^2y^2 - 2ab\operatorname{cosec}2\alpha\, xy = 0 \text{ is } 2ab(\cot 2\alpha + \operatorname{cosec}2\alpha\log\tan\alpha).$$

7. Show that the whole area between the curve

$$y^2(\alpha-x)(x-\beta) - k^2x^2 = 0,$$

and the lines $$x=\alpha, \quad x=\beta, \quad \text{is} \quad \pi k(\alpha+\beta).$$

8. Show that the whole area between the curve

$$y^2(\alpha-x)(x-\beta) - k^4 = 0,$$

and the lines $$x=\alpha, \quad x=\beta, \quad \text{is} \quad 2\pi k^2.$$

156. We now proceed to consider an important class of curves, namely, those called unicursal. A curve of this nature is such that the co-ordinates of any point on it can be

expressed rationally in terms of a parameter. It is evident then that the expression for the elementary area will be a rational differential, so that the areas of these curves can never involve higher transcendents than logarithms and circular functions.

In fact, suppose we take

$$x = \frac{f_1}{f}, \quad y = \frac{f_2}{f}, \tag{29}$$

where f_1, f_2, f are three polynomials in the variable θ of the n^{th} degree, or are homogeneous expressions in two variables λ, μ of the same degree, then the locus of the point xy is a curve of the n^{th} order; for if we seek the points where a line $\alpha x + \beta y + \gamma = 0$ meets the curve, we get $\alpha f_1 + \beta f_2 + \gamma f = 0$, that is, an equation of the n^{th} degree, to determine the points of intersection; and this shows that an arbitrary line meets the curve in n points. For the area then swept out by the radius vector from the origin, we find

$$dS = \tfrac{1}{2}(xdy - ydx) = \tfrac{1}{2} xyd \log\left(\frac{y}{x}\right)$$

$$= \frac{1}{2}\frac{f_1 f_2}{f^2} d\log\left(\frac{f_2}{f_1}\right) = \frac{f_1 df_2 - f_2 df_1}{2f^2};$$

but $$\lambda \frac{df_1}{d\lambda} + \mu \frac{df_1}{d\mu} = nf_1, \quad df_1 = \frac{df_1}{d\lambda} d\lambda + \frac{df_1}{d\mu} d\mu,$$

$$\lambda \frac{df_2}{d\lambda} + \mu \frac{df_2}{d\mu} = nf_2, \quad df_2 = \frac{df_2}{d\lambda} d\lambda + \frac{df_2}{d\mu} d\mu;$$

hence we get

$$dS = \frac{1}{2n}\left(\frac{df_1}{d\lambda}\frac{df_2}{d\mu} - \frac{df_1}{d\mu}\frac{df_2}{d\lambda}\right)\frac{(\lambda d\mu - \mu d\lambda)}{f^2};$$

so that, putting

$$\frac{df_1}{d\lambda}\frac{df_2}{d\mu} - \frac{df_1}{d\mu}\frac{df_2}{d\lambda} = n^2 J,$$

we obtain
$$S = \int \frac{nJ(\lambda d\mu - \mu d\lambda)}{2f^2}, \tag{30}$$

where we may take either λ or μ equal to unity, or, in fact, assume any relation between λ, μ, according to convenience.

In treatises on curves it is shown that a unicursal curve of the n^{th} degree has the greatest number of double points which a curve of that degree can have, and that in general there are two parameters corresponding to each double point. If, then, there is a loop of the curve terminated by a real double point, as in Fig. 10, we find its area by taking the integral (30) between the limits corresponding to the two parameters of the double point.

The unicursal curve of the third degree has one double point, and that of the fourth degree has three double points. Of these curves we have already considered some special forms among the examples, and in each case, as a verification of the preceding results, we may notice that the area involves no higher transcendents than logarithms or circular functions.

Examples.

1. To find the area of the loop of the quartic curve

$$x = \frac{\theta}{\theta^2 + a^2}, \quad y = \frac{\theta}{\theta^2 + b^2}.$$

In this case the loop has its corresponding node at the origin, the values of the parameter for that point being ∞, 0. We find then

$$dS = \left\{\frac{1}{\theta^2 + a^2} - \frac{1}{\theta^2 + b^2}\right\} \frac{\theta^3 d\theta}{(\theta^2 + a^2)(\theta^2 + b^2)},$$

whence we get

$$S = \frac{-1}{2(a^2-b^2)}\left\{\frac{a^2}{\theta^2+a^2}+\frac{b^2}{\theta^2+b^2}\right\}+\frac{(a^2+b^2)}{2(a^2-b^2)^2}\log\left(\frac{\theta^2+a^2}{\theta^2+b^2}\right)$$

as the general expression for the area. Hence, putting 0, ∞, successively for θ, we have for the area of the loop

$$S = \frac{a^2+b^2}{(a^2-b^2)^2}\log\left(\frac{a}{b}\right)-\frac{1}{a^2-b^2}.$$

2. Show that the whole area of the curve represented by the equations

$$x=\frac{\theta}{\theta^2+a^2},\quad y=\frac{1}{\theta^2+b^2}\quad\text{is}\quad\frac{\pi}{b(a+b)^2}.$$

3. A loop of the curve

$$x=\frac{\theta(a\theta^2+2b\theta+c)}{1+\theta^4},\quad y=\frac{\theta(a'\theta^2+2b'\theta+c')}{1+\theta^4}$$

has its node at the origin; show that its area is

$$\frac{\pi}{8\sqrt{2}}\{ab'-a'b+3(bc'-bc)\}+\frac{1}{4}(ac'-a'c).$$

157. There are several cases in which, when we generate a new curve from a given one by some geometrical method, we can find the expression for the area of the former curve in terms of the area of the latter, and some other simple expressions. For instance, suppose we transform a curve by substituting $y+\phi(x)$ for y, where $\phi(x)$ is some function of x; then, by the formula (1), the difference of the corresponding areas is

$$\int_{x_2}^{x_1}\phi(x)\,dx.$$

Hence, if we assume $\phi(x)$ so that this integral is known, the area of the generated curve can be found, when that of the given curve has been obtained. It may be noticed that if $\phi(x)$ is a rational function of x, and the given curve consist

of a closed oval, the areas of the two curves will be equal; for if y_1, y_2 are the values of y corresponding to the same value of x for the given curve, and y'_1, y'_2 the similar values for the generated curve, we have

$$y'_1 = y_1 + \phi(x), \quad y'_2 = y_2 + \phi(x);$$

hence
$$y'_1 - y'_2 = y_1 - y_2,$$

and
$$\int (y'_1 - y'_2)\, dx = \int (y_1 - y_2)\, dx.$$

Hence, from (4), and since the limiting values of x are evidently the same for both curves, the result stated above follows at once. Several of the examples already given are particular cases of the application of this method.

158. In the case of polar co-ordinates, we may notice a few transformations. If we measure out a constant length k on the radius vector from the point on a curve, we have

$$\Sigma = \tfrac{1}{2} \int (r \pm k)^2 d\theta$$

for the area of the locus of the extremity, the double sign corresponding to the two directions in which k can be measured. Hence we have, if Σ, S are the entire areas,

$$\Sigma - S = \pi k^2 \pm k \int_0^{2\pi} r\, d\theta,$$

the given curve being supposed to be an oval enclosing the origin. In this case, then, if Σ, Σ' correspond to the positive and negative values of k, we get

$$\Sigma + \Sigma' = 2S + 2\pi k^2.$$

Again, if, leaving θ unaltered, we put $\sqrt{(r^2 - k^2)}$ for r, we have

$$\Sigma = \tfrac{1}{2} \int (r^2 - k^2)\, d\theta = S - \tfrac{1}{2} k^2 (\theta_1 - \theta_2)$$

for a portion of the sectorial area of the generated curve.

Also, if we change θ into $n\theta$, and leave r unaltered, we have

$$\Sigma' = \tfrac{1}{2}\int \frac{r^2}{n} d\theta = \frac{1}{n}\Sigma;$$

that is, the area of the generated curve is equal to the n^{th} part of the corresponding area of the given curve.

Examples.

1. Show that the whole area bounded by the loops of the curve

$$(my + ax^2 + \beta x)^2 - x^2(a^2 - x^2) = 0 \quad \text{is} \quad 4a^3/3m.$$

2. Show that the sum of the areas of the two ovals of the curve

$$r^2 - 2(b + a\cos\theta)r + 2ab\cos\theta + c^2 = 0 \quad \text{is} \quad 2\pi(2b^2 + a^2 - c^2).$$

3. Show that the area of one of the ovals of the curve

$$r^2 - 2ar\cos 3\theta + a^2 - 9k^2 = 0 \quad \text{is} \quad \pi k^2.$$

4. If from a given oval curve we generate two other ovals, by putting $r \pm \sqrt{\phi'(\theta)}$ for r, and leaving θ unaltered, show that the sum of their sectorial areas, minus double the corresponding area of the given curve, is $\phi(\theta_1) - \phi(\theta_2)$, where θ_1, θ_2 determine the limiting radii vectores.

159. We consider here the expression for the area swept out by a radius vector round an origin, in the case in which the curve is given as the envelope of the line

$$x\cos\omega + y\sin\omega - p = 0.$$

To find the point of contact of this line, we have, by differentiating in accordance with the theory of envelopes,

$$x\sin\omega - y\cos\omega + \frac{dp}{d\omega} = 0,$$

$$\cos\omega\frac{dx}{d\omega} + \sin\omega\frac{dy}{d\omega} = 0.$$

Again, from the first of these equations, we get

$$\sin\omega\,\frac{dx}{d\omega} - \cos\omega\,\frac{dy}{d\omega} + x\cos\omega + y\sin\omega + \frac{d^2p}{d\omega^2} = 0,$$

or

$$\cos\omega\,\frac{dy}{d\omega} - \sin\omega\,\frac{dx}{d\omega} = p + \frac{d^2p}{d\omega^2}.$$

Now we have, identically,

$$\begin{aligned}\frac{xdy}{d\omega} - \frac{ydx}{d\omega} &= \left(x\cos\omega + y\sin\omega\right)\left(\cos\omega\,\frac{dy}{d\omega} - \sin\omega\,\frac{dx}{d\omega}\right)\\ &\quad + \left(x\sin\omega - y\cos\omega\right)\left(\cos\omega\,\frac{dx}{d\omega} + \sin\omega\,\frac{dy}{d\omega}\right)\\ &= p\left(p + \frac{d^2p}{d\omega^2}\right),\end{aligned}$$

from the equations given above. But there is

$$\frac{d}{d\omega}\left(\frac{pdp}{d\omega}\right) = \frac{pd^2p}{d\omega^2} + \left(\frac{dp}{d\omega}\right)^2,$$

so that we get

$$S = \frac{1}{2}\int(xdy - ydx) = \frac{1}{2}\frac{pdp}{d\omega} + \frac{1}{2}\int\left\{p^2 - \left(\frac{dp}{d\omega}\right)^2\right\}d\omega, \quad (31)$$

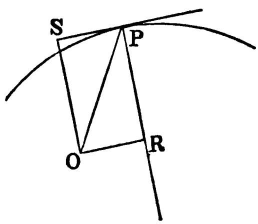

Fig. 22.

or $$S = OPS + \tfrac{1}{2}\int(OS^2 - OR^2)\,d\omega,$$

if, as in the figure, S, R are the feet of the perpendiculars

from O on the tangent, and normal, respectively, at the point P of the curve. By means of this result, we can find the area of a curve, when the perpendicular p is given as a function of ω. And it may be observed that, in general, we have for the area S of a closed oval curve, if we write q for OR,

$$S = \frac{1}{2}\int_0^{2\pi} (p^2 - q^2)\, d\omega, \tag{32}$$

as the area OPS vanishes between the limits.

160. As an example of the use of the formula (31), let us consider the curve parallel to a parabola, where, by the curve parallel to a given one, we mean the envelope of a line situated at a constant distance from the tangent of the given curve. The parallel is thus obtained by substituting $p \pm k$ for p, and leaving ω unaltered in the relation connecting p and ω for an assigned curve. Now, for the parabola referred to its focus as origin, we have $p = m \sec \omega$, and hence for the parallel curve, we get

$$p = m \sec \omega \pm k.$$

Applying, then, the formula (31), we have, for the area,

$$S = \frac{m \sin \omega}{2 \cos^2 \omega}\left(\frac{m}{\cos \omega} \pm k\right)$$

$$+ \frac{1}{2}\int \left\{ \frac{m^2}{\cos^2 \omega} - \frac{m^2 \sin^2 \omega}{\cos^4 \omega} \pm \frac{2\,km}{\cos \omega} + k^2 \right\} d\omega$$

$$= m^2 (\tan \omega + \tfrac{1}{3} \tan^3 \omega) \pm km \{\log (\sec \omega + \tan \omega) + \tfrac{1}{2} \sin \omega \sec^2 \omega\}$$

$$+ \tfrac{1}{2} k^2 \omega, \tag{33}$$

the constant being determined, so that S vanishes with ω.

If we write this expression for S in the form

$$A \pm Bk + Ck^2,$$

it is evident that A is the area bounded by the corresponding part of the arc of the parabola, and hence, that

$$Bk + Ck^2 - \tfrac{1}{2}kq, \quad Bk - Ck^2 - \tfrac{1}{2}kq,$$

are the values of the strips of area lying between the parabola and the two branches, respectively, of the parallel curve arising from the double sign of k. The difference of these strips is, therefore, $2\,Ck^2$ or $k^2\omega$, which is thus always finite, although each strip ultimately becomes infinite, when ω is made equal to $\pi/2$.

161. Again, as an example, let us consider the central *negative pedal* of an ellipse; that is, the curve enveloped by the perpendiculars erected to the central radii vectores at their extremities.

The ellipse being written in the usual form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have, for the negative pedal,

$$p = \frac{ab}{\sqrt{(a^2 \sin^2 \omega + b^2 \cos^2 \omega)}}.$$

Hence we get

$$\frac{dp}{d\omega} = -\frac{ab\,(a^2 - b^2) \sin \omega \cos \omega}{(a^2 \sin^2 \omega + b^2 \cos^2 \omega)^{\frac{3}{2}}};$$

and therefore, from (32), if S is the whole area,

$$S = \frac{a^2 b^2}{2} \int_0^{2\pi} \left\{ \frac{1}{a^2 \sin^2 \omega + b^2 \cos^2 \omega} - \frac{(a^2 - b^2)^2 \sin^2 \omega \cos^2 \omega}{(a^2 \sin^2 \omega + b^2 \cos^2 \omega)^3} \right\} d\omega.$$

This integral is most easily evaluated by the method of Art. 67, namely, by means of the substitution,

$$\cot\omega = a\cot\phi/b.$$

We thus get

$$S = \frac{1}{2}ab\int_0^{2\pi}\left\{1 - \frac{(a^2-b^2)^2}{a^2b^2}\sin^2\phi\cos^2\phi\right\}d\phi$$

$$= \frac{1}{2}ab\left\{2\pi - \frac{\pi(a^2-b^2)^2}{4a^2b^2}\right\}$$

$$= \frac{\pi}{8ab}(10a^2b^2 - a^4 - b^4). \qquad (34)$$

It is to be observed that, in this result, $2b^2$ is supposed to be greater than a^2, the curve being then as the inner oval in the figure.

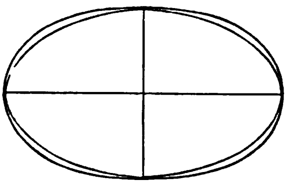

Fig. 23.

If this were not the case, the curve would have loops, and the value (34) would then express the difference of certain areas.

EXAMPLES.

1. Show from (32) that in general the whole area of the pedal of the evolute of a given closed curve, with regard to an internal point O, is equal to the area enclosed between the curve and its pedal with regard to O.

2. Show that the whole area of the pedal of the evolute of an ellipse is

$$\frac{\pi}{2}\{(a-b)^2+\delta^2\},$$

where a, b, are the semiaxes of the ellipse, and δ is the distance of the origin from the centre.

3. Show that the whole area of the curve

$$\{4(a^2+b^2-ab)-3(x^2+y^2)\}^3=\{9(2b-a)x^2+9(2a-b)y^2 \\ -4(a+b)(2a-b)(2b-a)\}^2,$$

where $2b>a$, is $\qquad \dfrac{\pi}{8}(10ab-a^2-b^2)$.

This curve is the envelope of the line

$$x\cos\omega+y\sin\omega=a\cos^2\omega+b\sin^2\omega.$$

4. Show that the whole area of the curve

$$(x^2+y^2)^2+8b(x^3-3xy^2)+18b^2(x^2+y^2)-27b^4=0 \text{ is } 2\pi b^2.$$

This curve is the envelope of the line

$$x\cos\omega+y\sin\omega=b\cos 3\omega.$$

5. Show that the difference of the areas lying between a closed oval curve and the two parts of the parallel curve on either side is $2\pi k^2$ where k is the constant distance from the parallel curve.

6. Show that the sum of the areas of the ovals formed by the curve parallel to the ellipse at the distance k is $2\pi(k^2+ab)$, where a, b are the semiaxes of the ellipse.

7. If S_1, S_2 are the whole areas of the pedals of a curve and its evolute with regard to a point O, show that the whole area of the locus of the feet of the perpendiculars from O on the lines inclined to the curve at a constant angle α is

$$S_1\cos^2\alpha+S_2\sin^2\alpha.$$

8. If S is the area lying between a curve and its negative pedal with regard to an internal point, show that, in general,

$$S = \frac{1}{2}\int_0^{2\pi} \left(\frac{dr}{d\theta}\right)^2 d\theta,$$

where r, θ are polar co-ordinates.

To find the values of S in the case of the ellipse, if $a\cos\phi$, $b\sin\phi$, are the co-ordinates of a point on the curve, and α, β those of the origin, we have

$$r^2 = (\alpha - a\cos\phi)^2 + (\beta - b\sin\phi)^2,$$

$$\tan\theta = \frac{b\sin\phi - \beta}{a\cos\phi - \alpha}.$$

Hence $$r\frac{dr}{d\phi} = a\alpha\sin\phi - b\beta\cos\phi - (a^2 - b^2)\sin\phi\cos\phi,$$

$$r^2\frac{d\theta}{d\phi} = ab - b\alpha\cos\phi - a\beta\sin\phi.$$

We thus obtain

$$S = \frac{1}{2}\int_0^{2\pi} \left(\frac{dr}{d\phi}\right)^2 \frac{d\phi}{d\theta}\, d\phi = \frac{1}{2}\int_0^{2\pi} \frac{\{a\alpha\sin\phi - b\beta\cos\phi - (a^2 - b^2)\sin\phi\cos\phi\}^2 d\phi}{ab - b\alpha\cos\phi - a\beta\sin\phi}.$$

The evaluation of this definite integral presents no difficulty, but the actual labour of doing so would be tedious. If $\beta = 0$, we have

$$S = \frac{1}{2b}\int_0^{2\pi} \frac{\sin^2\phi\,\{a\alpha - (a^2 - b^2)\cos\phi\}^2 d\phi}{a - \alpha\cos\phi},$$

which, by putting

$(l + m\cos\phi + n\cos^2\phi + p\cos^3\phi)(a - \alpha\cos\phi) + q$ for $\sin^2\phi\,\{a\alpha - (a^2 - b^2)\cos\phi\}^2$

gives

$$S = \frac{\pi}{2b}\left\{2l + n + \frac{2q}{\sqrt{(a^2 - \alpha^2)}}\right\}.$$

We then find the values of l, n, and q by equating the coefficients in the identity just written down; and we thus get finally

$$S = \frac{\pi a}{2b\alpha^4}\left\{2(a^2 + c^2)\alpha^4 - (c^4 + 4a^2c^2)\alpha^2 + 2a^2c^4 - 2a(a^2 - c^2)^2\sqrt{(a^2 - \alpha^2)}\right\},$$

where $$c^2 = a^2 - b^2.$$

9. Show that the whole area between the curve $r(a^2\cos^2\theta + b^2\sin^2\theta) = k^3$, and its negative pedal with regard to the origin is

$$\frac{\pi k^6(a^2-b^2)^2(a^2+b^2)}{4a^5b^5}.$$

10. Show that the whole area between the curve $kr = a^2\cos^2\theta + b^2\sin^2\theta$ and its negative pedal with regard to the origin is

$$\frac{\pi(a^2-b^2)^2}{2k^2}.$$

162. In connexion with the subject of pedals, we give here Steiner's relation connecting the areas of the pedals of a closed curve with regard to different internal points, O, O'. Let p, ϖ be the values of the perpendiculars for these origins, respectively, then we may write

$$S = \frac{1}{2}\int_0^{2\pi} p^2\,d\omega, \quad S' = \frac{1}{2}\int_0^{2\pi} \varpi^2\,d\omega;$$

but

$$\varpi = p - x\cos\omega - y\sin\omega,$$

where x, y are the co-ordinates of O' with regard to rectangular axes drawn through O. Hence we have

$$S' = \frac{1}{2}\int_0^{2\pi}\{x^2\cos^2\omega + y^2\sin^2\omega + 2xy\sin\omega\cos\omega$$
$$- 2p(x\cos\omega + y\sin\omega) + p^2\}\,d\omega$$
$$= \frac{\pi}{2}(x^2+y^2) - \alpha x - \beta y + S, \qquad (35)$$

where we have written α, β for the definite integrals

$$\int_0^{2\pi} p\cos\omega\,d\omega, \int_0^{2\pi} p\sin\omega\,d\omega, \text{ respectively.}$$

This result shows that when we are given the area for one origin, and the values of the corresponding definite integrals α, β, then we can at once obtain the area of the pedal for any other origin. Again, from (35), we see that if the area S' of the pedal of a closed curve with regard to a point O' be given, the locus of O' is a circle, whose centre is the same for all values of S'. If the origin O be taken at the centre of this circle, the constants α, β disappear. The area of the pedal is then a minimum, and the area of the pedal with regard to another origin is equal to the minimum area plus double the area of the circle described on the line joining the two origins as diameter.

163. If the given curve be not closed, or if we consider the area of a portion only of the pedals, the limits of integration will be different—say they are ω_1, ω_2. If then we write

$$\int_{\omega_2}^{\omega_1} \cos^2 \omega d\omega = a, \int_{\omega_2}^{\omega_1} \sin^2 \omega d\omega = b, \int_{\omega_2}^{\omega_1} \sin \omega \cos \omega d\omega = h,$$

$$\int_{\omega_2}^{\omega_1} p \cos \omega d\omega = \alpha, \int_{\omega_2}^{\omega_1} p \sin \omega d\omega = \beta,$$

we have

$$S' = \tfrac{1}{2}(ax^2 + by^2 + 2hxy) - \alpha x - \beta y + S, \qquad (36)$$

where S, S' are now the areas of portions of the pedals corresponding to the same limiting values of ω.

The locus which was a circle in the preceding case is thus replaced by a conic. This conic, it is easily shown, is an ellipse; for, if we take the actual values of a, b, h, we find

$$4(ab - h^2) = \{\omega_1 - \omega_2 + \sin(\omega_1 - \omega_2)\cos(\omega_1 + \omega_2)\}$$
$$\{\omega_1 - \omega_2 - \sin(\omega_1 - \omega_2)\cos(\omega_1 + \omega_2)\}$$
$$- \sin^2(\omega_1 - \omega_2)\sin^2(\omega_1 + \omega_2) = (\omega_1 - \omega_2)^2 - \sin^2(\omega_1 - \omega_2);$$

that is, $ab - h^2$ is always positive, and this, as is shown in treatises on conics, is the condition that (36) should represent an ellipse.

164. There are two interesting theorems of Steiner's concerning the connexion of the areas and arcs of roulettes with those of pedals. The theorem on the area is: If a closed curve rolls on a right line and makes a complete revolution, the area between the line and the locus of a point O rigidly connected with the rolling curve is equal to double the area of the pedal of O with respect to the curve.

To prove this, we remark first that we get from (31)

$$\frac{ds}{d\omega} = p + \frac{d^2p}{d\omega^2},$$

or

$$\frac{d(s-q)}{d\omega} = p, \tag{37}$$

by putting pds for $2dS$, where ds is the element of the arc of the curve.

Now, if P is the point of contact of the rolling curve in

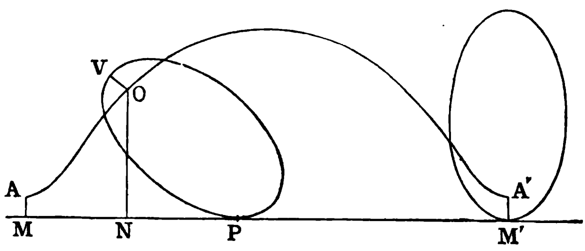

Fig. 24.

any position, and A, M; A', M', are the positions of O, V,

initially, and after a complete revolution, respectively, we must have

$$\text{arc } VP = MP = MN + NP, \quad \text{or } s - q = MN,$$

where

$$VP = s, \quad NP = q.$$

Therefore, from (37), we have $dx = pd\omega$, where x, y are the co-ordinates of O with regard to the areas MA, MN. Hence, since $y = p$, we get

$$ydx = p^2 d\omega;$$

and if

$$MM' = a,$$

$$\int_0^a ydx = \int_0^{2\pi} p^2 d\omega,$$

which is the statement in symbols of the theorem concerning the areas enunciated above. This theorem, combined with that of the preceding Article, will give several results concerning the areas of roulettes; for example, we infer that if a curve roll on a right line, there is one point connected with the curve for which the entire area of the corresponding roulette is a minimum; also that the area of the roulette corresponding to any other point exceeds this minimum area by the area of a circle whose radius is equal to the distance between the points.

EXAMPLES.

1. If the sum of the areas of the loops of the pedal of the point O with regard to the cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$ is a minimum, show that O is given by the equations $\theta = 0$, $r = \frac{1}{4} a$, and that the corresponding sum of the areas is $27 \pi a^2/64$.

2. If the sectorial area between the pedal of a closed curve and two radii vectores drawn through a point P parallel to given directions is a minimum, show that P is a determinate point, namely, the centre of the ellipse (36).

3. Show that the axes of the ellipse (36) are the bisectors of the angles between the bounding radii vectores of the areas.

4. Verify the expression for the area of the cycloid by means of Steiner's theorem on the areas of roulettes.

5. If an ellipse roll on a right line, show that the area of the roulette described by its focus in a complete revolution is double the area of the auxiliary circle.

6. If an ellipse roll on a right line, show that the area of the roulette described by any point O rigidly connected with it is equal to the sum of the areas of the director circle, and a circle whose radius is equal to the distance of O from the centre of the ellipse.

7. If a closed curve roll on another the same as itself so that corresponding points are always in contact, show that the area of the roulette generated by a point O is equal to four times the area of the pedal of O.

165. We now come to the consideration of several theorems concerning the areas swept out by lines which vary subject to certain conditions. For this purpose the fundamental theorem is the formula (6); the difference in this

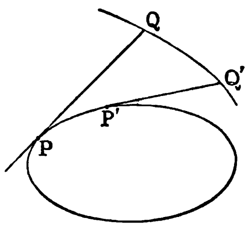

Fig. 25.

case being that O in Fig. 13 instead of being a fixed point is the ultimate intersection of the lines OP, OQ, that is, the point of contact of OP with its envelope. Integrating then between the limits α, β, we get, if P, P' are two points on the envelope,

$$\text{area } PP'QQ' = \frac{1}{2}\int_{\beta}^{\alpha} r'^2\, d\theta, \tag{38}$$

where the tangents PQ, $P'Q'$ in the figure correspond to the angles α, β, and the points Q, Q' lie on a curve which is defined by some relation connecting the length r of the tangent with the angle θ, which it makes with a fixed line.

166. For instance, if we measure out a constant length a on the tangent to a given curve, we find

$$\frac{a^2}{2}(\alpha - \beta)$$

as the expression for the area included between two tangents, the curve itself, and the generated locus. Hence the whole area between a closed curve and either branch of such a locus is πa^2. We say either branch, because there are two branches arising from the directions in which the length a may be measured. These two branches obviously intersect each other at certain points, from which the tangents drawn to the given curve have the same length. We see thus that the algebraic sum of the areas of the loops between the two branches is equal to zero.

167. Again, if we seek the area S included between a curve, two normals, and the evolute, we have, from (38),

$$S = \tfrac{1}{2}\int_\beta^\alpha \rho^2\, d\theta, \tag{39}$$

where ρ is the radius of curvature, and θ the angle turned through by the normal.

As an example, let us consider the ellipse; then the definite integral

$$\frac{1}{2}\int_0^{2\pi} \rho^2\, d\theta = \frac{1}{2}\int \rho\, ds = \frac{1}{2}\int_0^{2\pi} \frac{b'^3}{ab}\, b'\, d\phi = \frac{1}{2ab}\int_0^{2\pi} b'^4\, d\phi,$$

by known properties of the ellipse, where

$$b'^2 = a^2 \sin^2\phi + b^2 \cos^2\phi,$$

will evidently give the sum of the areas of the curve and its evolute. Hence, if S, Σ are these areas respectively, we have

$$S + \Sigma = \frac{1}{8ab}\int_0^{2\pi} \{a^2 + b^2 - (a^2 - b^2)\cos 2\phi\}^2 d\phi$$

$$= \frac{\pi}{4ab}\{(a^2 + b^2)^2 + \tfrac{1}{2}(a^2 - b^2)^2\},$$

from which, since $S = \pi ab$, we get

$$\Sigma = \frac{3\pi(a^2 - b^2)^2}{8ab}.$$

Examples.

1. If a length be measured out on the tangent to a given closed curve equal to the parallel radius vector from an internal point of another closed curve, show that the area between the first curve and either branch of the generated locus is equal to the area of the second curve.

2. If a length be measured out on the tangent to a given closed curve proportional to the square root of the radius of curvature, show that the area between the curve and the generated locus is proportional to the perimeter of the curve.

3. If the portion of the tangent of a closed curve intercepted on an outer curve is bisected at the point of contact, show that the area cut off from the outer curve is constant.

4. Show that the integral

$$\frac{3}{2}\int_{s_2}^{s_1} \frac{(s + k)^2 ds}{\sqrt{(4a^2 - s^2)}}$$

gives an expression for the area included between the cardioid

$$r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \tfrac{1}{2}\theta,$$

two tangents, and an involute.

168. We now give a general theorem concerning the areas intercepted by a pair of lines on two curves of the same degree which are connected in a certain manner. This may be stated as follows:—Let $\phi_n = 0$, $\phi_{n-3} = 0$, denote two general curves of the n^{th} and $(n - 3^{th})$ degrees, respectively; then, if a line meet the curves $\phi_n = 0$, $\phi_n + k\phi_{n-3} = 0$, in the points a, b, c, &c.; a', b', c', &c., respectively, and another line meet the same curves in the points α, β, γ, &c.; α', β', γ', &c., respectively, then the algebraic sum of all the areas $aa'\alpha\alpha'$, $bb'\beta\beta'$, &c., is equal to zero. To prove this, let the curves referred to an arbitrary origin be transformed to polar co-ordinates. We may write then

$$\phi_n = A_0 r^n + A_1 r^{n-1} + \ldots + A_n,$$

$$\phi_{n-3} = B_0 r^{n-3} + B_1 r^{n-4} + \ldots + B_{n-3};$$

and from these equations we see that if we seek the points where any radius vector meets the curves $\phi_n = 0$, $\phi_n + k\phi_{n-3} = 0$, the sum and the sum of the products in pairs of the roots of the determining equations are the same; for the coefficients of r^n, r^{n-1}, r^{n-2} are identical in each case.

Hence the sum of the squares of the roots also will be the same; so that we have $\Sigma r^2 = \Sigma r'^2$, which may be written

$$(r_1^2 - r_1'^2) + (r_2^2 - r_2'^2) + \ldots + (r_n^2 - r_n'^2) = 0,$$

where r_1, r_2, &c., are the roots of one equation, and r_1', r_2', &c., those of the other. Hence, multiplying the latter equation by $d\phi/2$, where ϕ is the angle through which the line turns, and integrating between the limits ϕ_1, ϕ_2, we get, from (38), the result stated above.

We may evidently connect in pairs the roots of one equation with those of the other in any order; but these various

modes of arrangement would merely give us different statements of the same result. If, however, k be taken sufficiently small, the two curves will be separated from each other by a small interval, and it will be then most convenient to connect the points which lie immediately adjacent to each other.

169. For example, let us consider a cubic, $\phi_3 = 0$, consisting, as in Fig. 26, of an oval and an infinite branch. The adjacent curve in this case is the cubic $\phi_3 + k = 0$, which is obviously a similar curve, having the same asymptote.

Drawing then two lines through the point O to meet the two curves in the points

$$A, B, C, \quad A', B', C'; \quad D, E, F, \quad D', E', F',$$

where the letters with accents refer to points on the outer curve, and those without accents to points on the inner, the

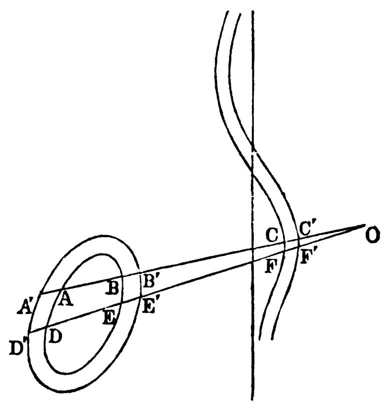

Fig. 26.

theorem given above becomes in this case

$$\text{area } A'ADD' = \text{area } BB'E'E + \text{area } CC'F'F.$$

Similarly, if the point O lies on the left side of the outer oval, or between the same oval and the infinite branch belonging to the inner, the difference of the areas intercepted between the ovals will be equal to the area intercepted between the infinite branches. Again, if O is interior to the inner oval, the sum of the areas intercepted between the ovals will be equal to the area intercepted between the infinite branches.

As a special case, let us consider a line touching the inner oval at O, and meeting the outer oval in A, B, and the infinite branches in C, C', respectively.

We have then

$$OA^2 + OB^2 + OC^2 = OC'^2;$$

whence

$$\int_0^{2\pi} (OA^2 + OB^2)\, d\phi = \int_0^{2\pi} (OC'^2 - OC^2)\, d\phi,$$

the interpretation of which is, it is easy to see—The area between the two ovals is equal to the area between the two infinite branches.

If the cubic have any other of its various forms, there is no difficulty in determining the corresponding relations between the areas.

170. Proceeding now to curves of the fourth order, we mention in particular the following result:—

Let $U = 0$ be a curve of the fourth order, consisting of two ovals, one within the other, and let $U + kL = 0$ be a similar curve, where L is a line which does not meet U in real points; then the areas of the two spaces between the ovals of these curves are equal. To prove this, let O be a point of contact of a tangent to the innermost oval; then if this tangent meet the other ovals in the points A, B, C, D;

A', B', where the accented and unaccented letters refer to

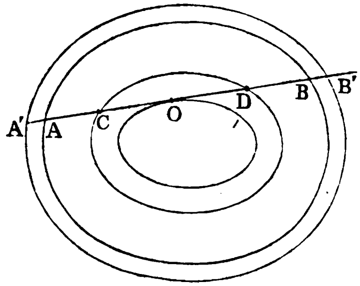

Fig. 27.

points on different curves, respectively, we have

$$OA'^2 + OB'^2 = OA^2 + OB^2 + OC^2 + OD^2,$$

from which we get

$$\int_0^{2\pi} (OA'^2 - OA^2 + OB'^2 - OB^2)\, d\phi = \int_0^{2\pi} (OC^2 + OD^2)\, d\phi;$$

but each of the integrals

$$\int_0^{2\pi} (OA'^2 - OA^2)\, d\phi, \quad \int_0^{2\pi} (OB'^2 - OB^2)\, d\phi$$

is equal to double the area between the outer ovals, while

$$\int_0^{2\pi} (OC^2 + OD^2)\, d\phi$$

is equal to four times the area between the two inner ones; so that the result we have stated above follows at once.

171. We now mention Mr. Holditch's theorem on areas. This theorem enables us to express the area of the curve described by a point P in terms of the areas of certain closed

curves described by two other points A, B, the point P being a fixed point on the line AB, which is of given length. To obtain the required expression, let $AP = a$, $PB = b$, and let

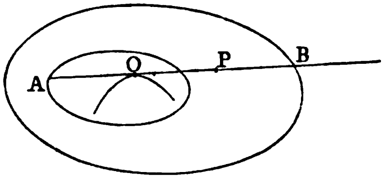

Fig. 28.

O be the point of contact of the line AB with its envelope; then if S, S_1, S_2, Σ are the areas of the curves described by the points P, A, B, O, respectively, we have

$$S_1 - \Sigma = \frac{1}{2}\int_0^{2\pi} OA^2\,d\phi = \frac{1}{2}\int_0^{2\pi}(r-a)^2\,d\phi,$$

$$S_2 - \Sigma = \frac{1}{2}\int_0^{2\pi} OB^2\,d\phi = \frac{1}{2}\int_0^{2\pi}(r+b)^2\,d\phi,$$

$$S - \Sigma = \frac{1}{2}\int_0^{2\pi} OP^2\,d\phi = \frac{1}{2}\int_0^{2\pi} r^2\,d\phi,$$

where $$OP = r.$$

Hence we get

$$aS_2 + bS_1 - (a+b)S = \frac{1}{2}\int_0^{2\pi}\{a(r+b)^2 + b(r-a)^2 - (a+b)r^2\}\,d\phi$$

$$= \pi ab(a+b);$$

or, $$S = \frac{aS_2 + bS_1 - \pi ab(a+b)}{a+b}. \qquad (40)$$

In this result the rigid line APB is supposed to return to its original position after making a complete revolution;

but if this be not the case, the formula (40) must be modified. Thus, if AB return to its original position after making n complete revolutions, we must substitute $n\pi$ for π in (40); and if it does not revolve at all, but returns after having reached a stationary position, we have $n = 0$, and we get

$$S = \frac{aS_2 + bS_1}{a + b}.$$

From the latter result we see that if the rod AB move with its ends on the same curve, or on two curves of equal area, and return to its original position without making a revolution, then any point P of the rod will describe a curve of equal area.

Again, if in the first case the ends A, B lie on the same curve, we have, from (40),

$$S_1 - S = \pi ab,$$

which result may be stated thus:—If a chord of given length move inside a closed curve, having a tracing point at the distances a, b from its ends, the area of the space between the two curves is equal to that of an ellipse whose semiaxes are a, b.

Examples.

1. Each of the cubics $\phi = 0$, $\phi + k = 0$, consists of an oval and an infinite branch: if it be possible to draw two lines through a point O so as to intercept equal areas between the ovals, show that O must lie between the infinite branches, or within the inner oval.

2. If the cubic $\phi = 0$ consists of an oval and an infinite branch, and the cubic $\phi + k = 0$ has a conjugate point within the oval, show that the area between the two infinite branches is equal to the area of the oval.

3. V is a cubic represented by $UL - k = 0$, where U is an ellipse and L a line. If A is the area of the oval of V, and A' the area between the infinite branch and the asymptote L; show that the difference of A, A' is equal to the area of the ellipse U.

4. If the cubic $L(x^2 + m^2 y^2) - k = 0$ has an oval, where L is a line, show that the area of the oval is equal to the area between the infinite branch and the asymptote L.

5. The axis of x meets the oval of the cubic $(x-a)(x^2+y^2) - b^3 = 0$ in A, B, the infinite branch in C, and the asymptote $x - a = 0$ in D. If the tangent at a point P of the oval meets the infinite branch and asymptote in Q, R, respectively, show that the area $QRCD$ is double the area $BPSO$, where O is the origin, and S is the foot of the perpendicular from O on the tangent at P.

6. The equations $S = 0$, $L = 0$ represent a circle and a line, respectively; show that the sum of the areas of the ovals of the Cartesian $S^2 - L = 0$ is equal to double the area of the circle S.

7. The equations $U = 0$, $V = 0$, $L = 0$ represent two ellipses and a line, respectively. If the quartic curve $UV - L = 0$ consists of two ovals, one wholly within the other, show that the sum of the areas of these ovals is equal to the sum of the areas of the ellipses.

8. The extremities A, B of a line of given length move on curves of equal area S, AB being supposed to return to its original position after a complete revolution. If Σ is the area of the curve described by a point P of the line AB, show that $S - \Sigma$ is equal to the area of the ellipse whose semiaxes are AP, PB.

9. If the extremities A, B of a line of given length return to their original positions on certain closed curves without completing circuits, while the line AB itself makes a complete revolution, show that the area of the curve described by a point P of the line AB is equal to the area of the ellipse whose semiaxes are AP, PB.

10. Tangents to a closed oval curve intersect at right angles in a point P; show that the whole area between the locus of P and the given curve is equal to half the area of the curve formed by drawing through a fixed point a radius vector parallel to either of the tangents and equal to the chord of contact.

11. A line of given length k moves with its ends A, B on a given closed curve; if O is its point of contact with its envelope, show that

$$\int_0^{2\pi} OA\,d\phi = \pi k,$$

where ϕ is the angle which AB makes with a fixed line.

12. A chord AB of the curve

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$$

subtends a right angle at the origin; show that the locus of the middle point of AB divides the area of the given curve into two equal parts.

172. It is usual, in treatises on the Integral Calculus, to give some account of the methods of approximating to the value of the area enclosed within a given boundary. The simplest way is to draw a sufficient number of parallel ordinates at equal intervals, and then consider the curve as coincident with the sides of the polygon formed by joining the extremities of the ordinates. Hence, if h is the common internal, and $y_0, y_1, \ldots y_n$, are the parallel ordinates, the area of the polygon is

$$h\{\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \ldots + y_{n-1}\};$$

for it evidently consists of a number of trapeziums of breadth h. This, then, is taken as the approximate expression for the area of the curve.

In order to get a more exact approximation, we may describe a curve of the form

$$y = a + bx + cx^2 + \ldots + gx^n,$$

through a certain number of points on the given curve, and then consider the area required as coincident with that of the curve whose equation we have just written down.

If we took $n + 1$ points on the given curve, we could evidently completely determine a parabolic curve of the degree n so as to pass through these points; but as the number of the points ought to be large, this process would in general be too troublesome. A simpler method is to break up the system of points into groups, through each of which we can describe a parabola of lower degree. Thus, if we take groups of three, the curve to be described through these points is the ordinary parabola

$$y = a + bx + cx^2.$$

In this case, if we take the intermediate ordinate as the axis of y, we have, for the area between the extreme ordinates,

$$\int_{-h}^{h} (a + bx + cx^2)\,dx = 2h\left(a + \tfrac{1}{3}ch^2\right);$$

but we have $\quad y_0 = a - bh + ch^2, \quad y_1 = a,$

$$y_2 = a + bh + ch^2;$$

therefore, $\quad y_0 + y_2 = 2y_1 + 2ch^2,$

so that the expression for the area becomes

$$\tfrac{1}{3}h(y_0 + 4y_1 + y_2).$$

If we suppose now that the number of points, $n + 1$, say, on the curve is odd, and add together the values of the parabolic areas, we get

$$\tfrac{1}{3}h\{y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\},$$

as an approximate expression for the area of the curve.

In the same way, by making use of parabolae of the third or higher degrees, we should obtain closer approximations to the value of any given area.

EXAMPLES.

1. If S is the sectorial area of the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{m}} + \left(\frac{y}{b}\right)^{\frac{n}{m}} = 1$$

about the origin, we find

$$S = \frac{mab}{2n}\int (t - t^2)^{\frac{m-n}{n}}\,dt,$$

by putting $\quad x = at^{\frac{m}{n}}, \quad y = b(1 - t)^{\frac{m}{n}}.$

Hence, if $n = 1$, S is algebraic; and if $n = 2$, it depends upon the elementary integrals.

2. If S is the sectorial area of the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{m}} + \left(\frac{y}{b}\right)^{\frac{1}{n}} = 1,$$

show that $$S = \frac{ab}{2}\int \{m + (n-m)t\}(1-t)^{n-1}t^{m-1}\,dt,$$

where $x = at^m$.

3. Show that the whole area between two branches of the curve

$$y^2(a^2 - x^2)(x^2 - b^2) - c^4x^2 = 0$$

and the lines $x = a$, $x = b$, is πc^2.

4. Show that the whole area of the curve

$$x^2y^2 - (a^n - x^n)(x^n - b^n) = 0$$

is $$\frac{\pi}{4n}\left(a^{\frac{n}{2}} - b^{\frac{n}{2}}\right)^2.$$

5. Show that the whole area between the curve

$$y^2(\alpha - x)(x - \beta) - (c^2 + m^2x^2)^2 = 0,$$

and the asymptotes $x = \alpha$, $x = \beta$,

is $$\frac{\pi}{4}\left\{m^2(3\alpha^2 + 3\beta^2 + 2\alpha\beta) + 8c^2\right\}.$$

6. Show that the whole area between the curve

$$y^2(a^n - x^n)(x^n - b^n) - k^4x^{2n-2} = 0$$

and the lines $x = a$, $x = b$, is $2\pi k^2/n$.

7. Show that the whole area between the curve

$$x^2y^2(a - x)(x - b) - c^6 = 0$$

and the lines $x = a$, $x = b$, is $\dfrac{2\pi c^3}{\sqrt{(ab)}}$.

8. If a curve be such that the length of the tangent, measured from the point of contact to a fixed line, is constant, a say; show that a portion of its area is equal to that of a circle of radius a.

Show also that the entire area between the infinite branches is equal to the entire area of the circle.

9. If x, y are the co-ordinates of a point on a closed curve, and ds the element of the arc, show that the entire area of the curve is equal to the integral

$$\int \frac{y\,dx}{ds}\,ds$$

taken throughout the entire perimeter.

10. Show that the area of a loop of the curve

$$r^m = a^m \cos m\theta \quad \text{is} \quad \frac{a^2\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{m+2}{2m}\right)}{\Gamma\left(\frac{1}{m}\right)}.$$

11. Show that the sectorial area of the spiral $r^2\theta = a^2$ included between the radii vectores corresponding to the angles $m\alpha$, α is $a^2 \log m/2$.

12. Show that the seetorial area of the spiral $r = a\theta$ included between the radii vectores corresponding to the angles $\alpha + \beta$, $\alpha - \beta$, is

$$\frac{a^2}{3}(\beta^3 - 3\beta\alpha^2).$$

13. If a curve be given by the equation

$$r = a + \frac{b}{\sin\phi},$$

where ϕ is the angle which the radius vector makes with the tangent; show that, if S is the sectorial area, measured from the point for which $\phi = \pi/2$,

$$S = \frac{b^2}{2}\cot\phi - \frac{ab}{2}\log\tan\frac{\phi}{2}.$$

14. Show that the whole area between the curve

$$y = ae^{-\frac{x^2}{b^2}}$$

and the axis of x is $ab\sqrt{\pi}$.

15. Show that the whole area between the curve

$$y = ae^{-x^2-\frac{c^2}{x^2}}$$

and the axis of x is $a\sqrt{\pi}e^{-2c}$.

16. If S' is the area included between two rectangular radii vectores and the Cassinian

$$r^4 - 2c^2r^2 \cos 2\theta - k^4 = 0,$$

and S the area included between the axes and the curve; show that

$$S' = S + c^2 \sin\theta \cos\theta.$$

17. If two radii vectores of the Cassinian

$$r^4 - 2c^2r^2 \cos 2\theta - k^4 = 0$$

be drawn, corresponding to two roots of the equation

$$\sqrt{(c^4 + k^4)} \cos 4\theta = c^2 \sin 2\theta\,;$$

show that the area included between them and the curve is equal to that between the axes and the curve.

18. If a unicursal curve of the n^{th} degree be touched by the line at infinity in n consecutive points, show that its area can be always expressed algebraically.

19. Show that the sectorial area bounded by two radii vectores r, r' of the parabola $r^{\frac{1}{2}} \cos \frac{1}{2}\theta = m^{\frac{1}{2}}$, and the curve is equal to

$$\frac{m^{\frac{1}{2}}}{3}\left\{\left(\frac{r + r' + \delta}{2}\right)^{\frac{3}{2}} - \left(\frac{r + r' - \delta}{2}\right)^{\frac{3}{2}}\right\},$$

where δ is the chord of the arc.

20. If S is the area between a closed curve and the envelope of a line making a constant angle α with it, and S' is the area between the same curve and its evolute, show that $S = S' \sin^2\alpha$.

21. A curve of the n^{th} degree has n conchoidal asymptotes (that is, is such that all of its intersections with the line of infinity are points of inflexion). Show that the algebraic sum of the areas included between two lines, the curve, and its asymptotes, is equal to zero.

22. Show that the area of a loop of the curve $r^m = a^m \cos m\theta$, and that of the pedal with regard to the origin, are in a constant numerical ratio.

23. To prove Kempe's theorem, namely: If one plane slide upon another and return to its original position, the locus of points in the moving plane which describe curves of equal area is a circle.

To prove this, let x, y; α, β, be two points in the moving plane, and let x', y' be the co-ordinates of x, y with regard to rectangular axes passing through α, β, and fixed in the moving plane, then we may write

$$x = \alpha + x' \cos\phi + y' \sin\phi, \quad y = \beta + x' \sin\phi - y' \cos\phi,$$

where ϕ is the angle between the axes $x'y'$ and xy. Hence we have

$$xdy - ydx = \alpha d\beta - \beta d\alpha + (d\beta + \alpha d\phi)(x' \cos\phi + y' \sin\phi)$$
$$- (d\alpha - \beta d\phi)(x' \sin\phi - y' \cos\phi) + (x'^2 + y'^2) d\phi.$$

If the body therefore return to its original position after making n revolutions, we get, by integration, an expression of the form

$$2S = 2S' + Lx' + My' + 2n\pi(x'^2 + y'^2),$$

where S, S' are the areas described by x, y and α, β, respectively. If we take the origin at the centre of the circle we must have $L = M = 0$, and we see thus that the circles obtained by varying the constant S are all concentric.

24. In the same case as in the preceding example, all lines which envelope roulettes of the same area are tangents to the same conic.

To prove this, $x' \cos\omega + y' \sin\omega - \varpi = 0$ obviously represents any line rigidly connected with the axes x', y', if ϖ and ω are given. Putting then for x', y' in terms of x, y from the preceding example, this line becomes

$$(x - \alpha)\cos(\phi - \omega) + (y - \beta)\sin(\phi - \omega) - \varpi = 0.$$

Hence, if p is the perpendicular from the origin,

$$p = \alpha \cos(\phi - \omega) + \beta \sin(\phi - \omega) + \varpi,$$

and $$q = \frac{dp}{d\phi} = -\alpha \sin(\phi - \omega) + \beta \cos(\phi - \omega) + \frac{d\alpha}{d\phi}\cos(\phi - \omega) + \frac{d\beta}{d\phi}\sin(\phi - \omega),$$

so that if S is the area of the envelope, we get from (32), a result of the form

$$2S = \int_0^{2n\pi} (p^2 - q^2)\, d\omega = C\varpi^2 + 2\varpi(G\cos\omega + F\sin\omega)$$
$$+ A\cos^2\omega + B\sin^2\omega + 2H\sin\omega\cos\omega,$$

which, if S is a constant, is known to be the condition that the line should touch a conic.

25. Show that all the conics in the preceding example corresponding to different values of S are confocal, and that their common centre coincides with the centre of Kempe's circles.—(Mr. W. S. M'Cay.)

CHAPTER VIII.

RECTIFICATION OF PLANE CURVES.

173. In this chapter we consider the rectification of curves, that is, the finding of their lengths, the word *rectification* having reference to the determination of a portion of a *right* line equal in length to an arc of a given curve.

In every case we may consider an element of the curve as coinciding in the limit with the right line joining two consecutive points when the distance between these points is indefinitely diminished. Hence, in rectangular Cartesian coordinates, since the length of the line joining the point x, y to the consecutive point $x + dx$, $y + dy$ is

$$\sqrt{(dx^2 + dy^2)},$$

we may write

$$ds = \sqrt{(dx^2 + dy^2)},$$

where s is the length of the curve measured from a fixed point on it.

Integrating, then, we get

$$s = \int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx, \tag{1}$$

where y is supposed to be expressed in terms of x; but if x is given as a function of y, the formula to be used is

$$s = \int \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy.$$

More generally, if x and y are both given as functions of another variable θ, we have

$$s = \int \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}\, d\theta. \tag{2}$$

If the axes of co-ordinates were oblique, containing an angle ω, instead (1) we should have

$$s = \int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2 + 2\cos\omega\,\frac{dy}{dx}\right\}}\, dx, \tag{3}$$

and similar formulæ in the other cases.

In all the preceding integrals the limits of the independent variable must, of course, correspond to the extremities of the arc whose length is sought.

174. The simplest curve to which (1) can be applied is, perhaps, the circle. In this case we have

$$x^2 + y^2 = a^2, \quad \text{whence} \quad y = \sqrt{(a^2 - x^2)},$$

and

$$\frac{dy}{dx} = \frac{-x}{\sqrt{(a^2 - x^2)}}. \tag{4}$$

Hence

$$s = \int \sqrt{\left(1 + \frac{x^2}{a^2 - x^2}\right)}\, dx = \int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a},$$

if the arc is supposed to be measured from the axis of y.

Or thus: putting

$$x = a\cos\theta, \quad y = a\sin\theta,$$

we get, from (2),

$$s = \int a\,d\theta = a\theta,$$

as we know already.

175. Proceeding now to the case of the parabola, which we write in the form $y^2 = 4ax$,

we have $$\frac{dx}{dy} = \frac{y}{2a},$$

and, therefore, $$s = \int \sqrt{\left\{1 + \frac{y^2}{4a^2}\right\}}\, dy$$

$$= \frac{y\sqrt{(y^2 + 4a^2)}}{4a} + a \log\left\{\frac{y + \sqrt{(y^2 + 4a^2)}}{2a}\right\}, \qquad (5)$$

if the arc be measured from the vertex of the curve.

It may be observed that the first part of this expression for s, namely,

$$\frac{y}{4a}\sqrt{(y^2 + 4a^2)},$$

is equal to the length of the tangent measured from the point of contact to the axis of y, so that the difference of the arc and this portion of the tangent is represented by a pure logarithmic function.

176. Proceeding to the case of the ellipse, we may take

$$x = a \sin\phi,$$

if the curve is supposed to be written in the usual form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

We have then $$y = b\cos\phi,$$

and $$\left(\frac{ds}{d\phi}\right)^2 = a^2\cos^2\phi + b^2\sin^2\phi.$$

Hence, if we measure the arc from the extremity of the axis minor, we have

$$s = \int \sqrt{(a^2 \cos^2\phi + b^2 \sin^2\phi)}\, d\phi = aE(\phi), \qquad (6)$$

where the modulus of the elliptic integral is equal to the eccentricity of the ellipse.

If we integrate between the limits 2π and 0, we find that the whole perimeter of the ellipse is equal to $4aE$.

177. In the case of the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

we may take $\qquad x = a \operatorname{cosec} \phi, \quad y = b \cot \phi.$

We have then

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{a^2 \cos^2\phi}{\sin^4\phi} + \frac{b^2}{\sin^4\phi};$$

therefore

$$s = -\int \sqrt{(b^2 + a^2 \cos^2\phi)}\, \frac{d\phi}{\sin^2\phi},$$

where we take the negative sign for convenience. Hence, integrating by parts, we obtain

$$\begin{aligned} s &= \cot\phi \sqrt{(b^2 + a^2 \cos^2\phi)} + \int \frac{a^2 \cos^2\phi\, d\phi}{\sqrt{(b^2 + a^2 \cos^2\phi)}} \\ &= c \cot\phi\, \Delta(\phi) + c\,\{E(\phi) - k'^2 E(\phi)\} \\ &\qquad\qquad - c\,(E - k'^2 K), \qquad (7) \end{aligned}$$

where $\qquad a^2 + b^2 = c^2, \quad a = kc,$

and the constant is taken so that s vanishes at the vertex of the curve.

This is, however, not the best method of rectifying the hyperbola. We shall obtain a simpler expression further on by means of a different formula.

Since the arcs of the conic sections are thus expressible by means of the elliptic integrals of the first and second kinds, we see, by the comparison theory of these functions, that there ought to be certain relations connecting the arcs with the lengths of lines. These relations are most readily obtained geometrically; and to their consideration we propose to devote a subsequent part of this chapter.

178. As an example of a curve of the third degree, let us consider the cissoid

$$(a - x)y^2 = x^3.$$

We have then
$$y = \frac{x^{\frac{3}{2}}}{\sqrt{(a - x)}};$$

whence
$$\frac{dy}{dx} = \frac{(3a - 2x)\sqrt{x}}{2(a - x)^{\frac{3}{2}}}.$$

Therefore, from (1) we get

$$s = \int \sqrt{\left\{1 + \frac{x(3a - 2x)^2}{4(a - x)^3}\right\}} dx = \frac{a}{2}\int \frac{\sqrt{(4a - 3x)}}{(a - x)^{\frac{3}{2}}} dx$$

Putting, then,
$$4a - 3x = (a - x)z^2,$$

we have
$$\frac{a}{a - x} = z^2 - 3;$$

whence
$$\frac{dx}{a - x} = \frac{2z dz}{z^2 - 3};$$

so that we get

$$s = \frac{a}{2}\int\sqrt{\left(\frac{4a-3x}{a-x}\right)}\frac{dx}{a-x} = a\int\frac{z^2\,dz}{z^2-3}$$

$$= az + \frac{a\sqrt{3}}{2}\log\left(\frac{z-\sqrt{3}}{z+\sqrt{3}}\right) + C. \tag{8}$$

Now, if the arc be measured from the origin, since $x = 0$, and therefore $z = 2$ for this point, we get

$$C = a\sqrt{3}\log(2+\sqrt{3}) - 2a.$$

Suppose we take x nearly equal to a, which obviously corresponds to a point at a great distance from the origin, we have

$$a = (a-x)z^2,$$

and from the equation of the curve

$$(a-x)y^2 = a^3,$$

approximately. Hence $az = y$; and from (8), putting $x = a$, or $z = \infty$, we get ultimately $s - y = C$. But when $x = a$, y is evidently the length of the asymptote measured from the axis of x, so that we deduce the following result:—The difference between the whole length of the cissoid

$$(a-x)y^2 = x^3 \quad \text{and its asymptote} \quad x = a$$

is a finite quantity equal to

$$2a\sqrt{3}\log(2+\sqrt{3}) - 4a.$$

179. Again, as a further example, let us take the semicubical parabola, which we write in the form

$$9ay^2 = 4x^3.$$

We have then $$y = \frac{2}{3}\frac{x^{\frac{3}{2}}}{\sqrt{a}};$$

whence $$\frac{dy}{dx} = \sqrt{\left(\frac{x}{a}\right)};$$

therefore $$s = \int (a + x)^{\frac{1}{2}} \frac{dx}{\sqrt{a}} = \frac{2}{3\sqrt{a}} \{(a + x)^{\frac{3}{2}} - a^{\frac{3}{2}}\},$$

if the arc is measured from the origin.

Examples.

1. The axes of co-ordinates being supposed to contain an angle ω, show that the arc of the curve $9ay^2 = 4x^3$, measured from the origin, is equal to

$$\frac{a\Delta}{3}(2\theta^2 + 2\theta\cos\omega + 2 - 3\cos^2\omega) - a\sin^2\omega\cos\omega\log\left\{\frac{\theta + \cos\omega + \Delta}{1 + \cos\omega}\right\},$$

where $$x = a\theta^2, \quad \Delta = \sqrt{(1 + \theta^2 + 2\theta\cos\omega)}.$$

2. If the arc s of the cubic $3a^2y = x^3$ be measured from the origin, show that $$3s = t + aF_k(\theta),$$

where t is the length of the tangent measured to the axis of y, and

$$x = a\tan\tfrac{1}{2}\theta, \quad k^2 = \tfrac{1}{2}.$$

3. If s is the arc of the curve $4a^3y = x^4$ measured from the origin, show that

$$4s = t + 3a^4\int_0^x \frac{dx}{\sqrt{(a^6 + x^6)}},$$

where t is the length of the tangent measured to the axis of y.

4. Show that the arc of the curve $ay^3 = x^4$, where the axes are oblique, can be expressed by means of algebraic and logarithmic expressions.

5. Show that the arc of the curve

$$(2x + a)^3 = 27a(x^2 + y^2)$$

is equal to $$\frac{(4x + 5a)}{12}\left\{\frac{8x + a}{3a}\right\}^{\frac{1}{2}},$$

if it be measured from the point

$$y = 0, \quad 8x = -a.$$

6. Show that the arc of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1,$$

measured from the point $x = a, \quad y = 0,$

is
$$\frac{a^3 - (a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{3}{2}}}{a^2 - b^2},$$

where $x = a\cos^3\phi, \quad y = b\sin^3\phi.$

7. Show that the arc of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

measured from the point

$$x = y = a/4\sqrt{2},$$

is $\dfrac{5a}{16}\cos 2\theta\sqrt{(1 + 3\cos^2 2\theta)} + \dfrac{5a}{16\sqrt{3}}\log\{\sqrt{3}\cos 2\theta + \sqrt{(1 + 3\cos^2 2\theta)}\},$

where $x = a\cos^5\theta.$

8. To find the length of the curve

$$8a^2y^2 = x^2(a^2 - x^2),$$

we put $x = a\sin\theta.$

We get then $2y\sqrt{2} = a\sin\theta\cos\theta;$

hence
$$\left(\frac{ds}{d\theta}\right)^2 = a^2\sin^2\theta + \frac{a^2}{8}\cos^2 2\theta = \frac{a^2}{8}(2 + \cos 2\theta^2),$$

and
$$s = \frac{a}{2\sqrt{2}}\int(2 + \cos 2\theta)\,d\theta = \frac{a}{2\sqrt{2}}(2\theta + \sin\theta\cos\theta),$$

the arc being measured from the origin. We thus find that the perimeter of one of the loops is $\pi a/\sqrt{2}$.

9. Let s be the arc of the curve

$$2(n^2 - 1)\frac{y}{x} = n(x^n + x^{-n}) - x^{-n} + x^n;$$

then we find

$$s = \frac{x}{2(n^2 - 1)}\{n(x^n - x^{-n}) - x^n - x^{-n}\} + C.$$

If $n^2 < 1$, we may take $C = 0$, if s is measured from the origin; but if $n^2 > 1$, we have $C = 1\,(n^2 - 1)$, if s is measured from the point $x = 1$, that is, one of the points at which the tangent is parallel to the axis of x.

10. If the arc s of the cubic $3x^2 = y(y-1)^2$ be measured from the origin, show that

$$s^2 = x^2 + \frac{4}{3}y^2.$$

11. Show that the arc of the curve

$$2y = \frac{1}{5}x^5 + ax^2 - \frac{a^2}{x} + \frac{1}{3}\frac{1}{x^3 + a}$$

is expressible as an algebraic function of x.

12. If a curve be expressed by means of the equations

$$y = \log(1 + \theta^2), \quad x = 2\tan^{-1}\theta - \theta,$$

show that θ is the arc measured from the origin.

13. If the arc s of the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$$

be measured from the vertex, show that

$$s^2 = y^2 - c^2.$$

180. To find the expression for the arc of a curve in polar co-ordinates, we put $r\cos\theta$, $r\sin\theta$, for x, y, respectively, in the equation $ds^2 = dx^2 + dy^2$. Since we have then

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta, \quad dy = \sin\theta\,dr + r\cos\theta\,d\theta,$$

we obtain

$$ds^2 = dr^2 + r^2 d\theta^2.$$

This result is also evident at once from Fig. 13; for the element of the curve is the diagonal of a rectangle whose sides are dr, $r\,d\theta$.

Hence we get

$$s = \int\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta, \quad \text{or} \quad \int\sqrt{\left\{1 + \left(\frac{r\,d\theta}{dr}\right)^2\right\}}\,dr, \qquad (9)$$

according as we take θ or r as the independent variable.

As an example, let us consider a circle referred to a point on itself. Taking the tangent at the origin as the initial line, we have $r = 2a \sin\theta$. Hence, since

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2,$$

we get $s = 2a\int d\theta = 2a\theta$, as we know otherwise.

181. Again, let us consider the lemniscate $r^2 = a^2 \cos 2\theta$. We have then

$$\frac{r\,dr}{d\theta} = -a^2 \sin 2\theta;$$

hence
$$s = a^2 \int \frac{d\theta}{\sqrt{(\cos 2\theta)}} = \frac{a^2}{\sqrt{2}} F(\phi), \tag{10}$$

where $\cos 2\theta = \cos^2\phi$, $k^2 = 1/2$, and the arc of the loop is measured from the vertex.

Examples.

1. Show that the formula for the element of the arc of a curve in polar coordinates can be obtained from the expression for the distance between two points.

2. Show that the arc of the curve $r^m = a^m \cos m\theta$ is given by the equation

$$s = \int \frac{a^m\,dr}{\sqrt{(a^{2m} - r^{2m})}}.$$

3. If the arc s of the cardioid $r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \frac{1}{2}\theta$ is measured from the point for which $\theta = 0$, show that $s^2 = 4a(a - r)$.

4. Show that the arc s of the curve,

$$r = a\sqrt{(n^2 - 1)} + na \cos\frac{\theta}{n},$$

is given by the equation

$$s = na\theta + na\sqrt{(n^2 - 1)} \sin\frac{\theta}{n}.$$

5. Show that an arc of the curve $r = a \sin n\theta$ is equal in length to that of an ellipse whose semiaxes are a, na.

6. Show that an arc of the limaçon $r = a + b\cos\theta$ is equal in length to an arc of an ellipse whose axes are $a + b$, $a - b$.

7. Show that the length of an arc of the logarithmic spiral is proportional to the difference of the radii vectores of its extremities.

8. If the arc s of the curve $r\theta = a$ be measured from the point for which $r = a$, show that

$$s = a\sqrt{2} - \frac{a}{\theta}\sqrt{(1+\theta^2)} + a\log\left\{\frac{\theta + \sqrt{(1+\theta^2)}}{1+\sqrt{2}}\right\}.$$

182. We now mention Legendre's formula for the rectification of plane curves. In this case the curve is supposed to be given as the envelope of the line

$$x\cos\omega + y\sin\omega - p = 0,$$

and the formula sought is, in fact (37), already arrived at in Art. 164, namely,

$$\frac{d(s-q)}{d\omega} = p,$$

whence we get

$$s - q = \int p\,d\omega. \qquad (11)$$

This result is obviously of considerable use whenever the relation between p and ω assumes a simple form. It will also serve to give a geometrical representation of any proposed integral; for if the integral be brought to the form $\int p\,d\omega$ by some substitution, which can evidently be done in all cases, it is at once represented by the difference between the arc and a portion of the tangent belonging to a certain curve. If then from the proposed integral we have $p = \phi(\omega)$, the curve will be defined by the equations

$$x\cos\omega + y\sin\omega - p = 0, \quad x\sin\omega - y\cos\omega + \frac{dp}{d\omega} = 0,$$

or, if we put $\phi(\omega)$ for p and solve for x and y,

$$x = \phi(\omega) \cos \omega - \phi'(\omega) \sin \omega,$$

$$y = \phi(\omega) \sin \omega + \phi'(\omega) \cos \omega. \tag{12}$$

For example, let us consider the parabola referred to its focus. We have then $p \cos \omega = a$; hence we get

$$s - q = a \int \frac{d\omega}{\cos \omega} = a \log (\sec \omega + \tan \omega),$$

if the arc be measured from the vertex for which $\omega = 0$. This result is easily seen to be identical with that already obtained in Art. 175.

183. From (11) we can readily deduce the known result concerning the rectification of evolutes, namely, that any portion of the arc of the evolute of a given curve is equal to the difference of the radii of curvature corresponding to its extremities.

If the given curve be given as the envelope of the line $x \cos \omega + y \sin \omega - p = 0$, the evolute is the envelope of the normal, namely, $x \sin \omega - y \cos \omega + q = 0$. Hence for the arc of the evolute, we have from (11),

$$s = \frac{dq}{d\omega} + \int q\, d\omega = p + \frac{d^2 p}{d\omega^2} + C,$$

putting $dp/d\omega$ for q. Now, if ρ is the radius of the given curve,

$$\rho = \frac{ds}{d\omega} = p + \frac{d^2 p}{d\omega^2};$$

hence the result stated above follows at once.

184. We have already noticed, in Art. 160, how we may generate from a given curve another curve called the parallel. It is obtained by substituting $p \pm k$ for p, and leaving ω unaltered in the relation between p and ω for the given curve. Hence, for its arc s' we have, from (11),

$$s' = \frac{dp}{d\omega} + \int (p \pm k)\, d\omega.$$

If then s is the arc of the given curve, the difference $s' - s$ of corresponding portions is equal to

$$\pm \int k\, d\omega = \pm\, k\,(\omega_1 - \omega_2),$$

that is, the difference of the lengths of two corresponding arcs varies as the angle between the tangents at the extremities of either.

EXAMPLES.

1. If a closed curve without cusps be given as the envelope of the line

$$x \cos \omega + y \sin \omega - p = 0,$$

show that its entire perimeter is equal to

$$\int_0^{2\pi} p d\omega.$$

2. Show that the entire perimeter of the tricuspidal hypocycloid, namely, the envelope of

$$x \cos \omega + y \sin \omega = b \cos 3\omega \text{ is } 16b.$$

3. Show that the entire perimeter of the curve whose equation is given in Ex. 3, p. 265, is $\pi(a + b)$.

4. Show that any integral whose differential is algebraic can be represented by the difference of the arc and a portion of the tangent of an algebraic curve.

5. Show that the rectification of the curves

$$ay^2 = x^3, \text{ and } \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1$$

can be at once effected from the fact that they are the evolutes of known curves.

6. Show that the entire perimeter of the central negative pedal of an ellipse is $4bK$, where the modulus of the elliptic integral is equal to the eccentricity of the ellipse.

This readily follows from the fact that the perimeter of the negative pedal of a closed curve, with regard to an internal point, is equal to $\int_0^{2\pi} r d\theta$, where r, θ are polar co-ordinates with regard to the point.

7. Show that the entire perimeter of the negative pedal of an ellipse, with regard to a focus, is equal to $2\pi b$.

8. Show that the difference of the perimeters of a closed curve and one branch of a parallel curve is equal to $2\pi k$, where k is the constant interval between the curves.

9. C is a line midway between two parallel lines A, B; show that an arc of the curve touched by C is equal to half the sum of corresponding arcs of the curves touched by A and B.

185. We commence the investigation of the geometrical properties of the arcs of conic sections with the theorem known as Fagnani's, although it is but a particular case of the more general theorem of Dr. Graves.

Writing the equation of the ellipse referred to its centre in the usual manner, we have

$$p^2 = a^2 \cos^2 \omega + b^2 \sin^2 \omega.$$

Hence the formula (11) gives us

$$\text{arc } AP + PN = \int_0^\omega \sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)}\, d\omega, \qquad (13)$$

where the arc AP is measured from the vertex A, and it is to be observed that the portion PN of the tangent is taken with a negative sign.

But we have already proved, in Art. 176, that if an arc BP' of the curve be measured from the vertex B, then we have

$$\text{arc } BP' = \int_0^\phi \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)}\, d\phi.$$

It follows, hence, that if P, P' are two points on the curve, such that the angle ω, which determines the position of P, is

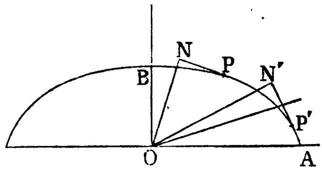

Fig. 29.

equal to the angle ϕ, which determines that of P', these angles belonging to different parametrical systems, then we have the relation

$$\text{arc } BP' = \text{arc } AP + PN,$$

or $$BP - AP' = PN. \tag{14}$$

It may be observed that the relation connecting the points P, P' is reciprocal, so that we may put $P'N'$ for PN; for, if ψ determines the position of P in the same way as ϕ does that of P', we have $\cot \psi = b \tan \omega / a$; hence, from $\phi = \omega$, we get

$$\cot \phi \cot \psi = b/a,$$

which is symmetrical with regard to ϕ and ψ. This result is consistent with a property of elliptic integrals of the second kind; for we proved, in the chapter on that subject, that if

$$F(\phi) + F(\psi) = K;$$

then $$\cot \phi \cot \psi = k',$$

and $$E(\phi) + E(\psi) = E + k^2 \sin \phi \sin \psi.$$

The latter relation, being written in the form

$$aE(\phi) - a\{E - E(\psi)\} = ak^2 \sin \phi \sin \psi,$$

is evidently equivalent to (14). The mode of investigation,

however, which we have adopted above has the advantage of giving an immediate geometrical signification to the algebraic part involved in the comparison of the integrals.

If the points P, P' coincide at Q, we have $\cot^2\phi = b/a$, and then

$$PN^2 = OQ^2 - ON^2 = a^2\sin^2\phi + b^2\cos^2\phi - (a^2\cos^2\phi + b^2\sin^2\phi)$$
$$= \frac{(a^2 - b^2)(1 - \cot^2\phi)}{1 + \cot^2\phi} = (a - b)^2.$$

Hence $PN = a - b$, and the arc AB is divided at the point Q, so that the difference of the parts is equal to the difference of the semiaxes.

It may be observed that the co-ordinates of the point Q are found to be

$$x = \frac{a^{\frac{3}{2}}}{\sqrt{(a + b)}}, \quad y = \frac{b^{\frac{3}{2}}}{\sqrt{(a + b)}}.$$

186. Applying the formula (11) to the hyperbola, in which case we write $p^2 = a^2\cos\omega - b^2\sin^2\omega$, we get

$$PN - \text{arc } AP = \int_0^\omega \sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}\,d\omega; \quad (15)$$

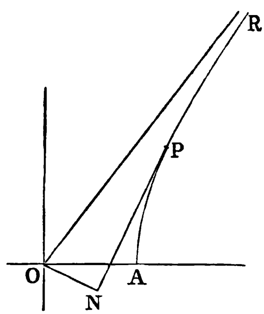

Fig. 30.

for the angle $OAN = \omega$ is measured on the negative side of

the axis OA. To reduce this expression to the standard forms of elliptic integrals, we get

$$\sqrt{(a^2+b^2)}\sin\omega = a\sin\theta.$$

We find, then,

$$\text{arc } AP = PN - \int\frac{ck^2\cos^2\theta\,d\theta}{\Delta(\theta)}$$

$$= PN - c\{E(\theta) - k'^2F(\theta)\}, \qquad (16)$$

where

$$\sqrt{(a^2+b^2)} = c, \quad a = kc, \quad \text{and} \quad PN = c\tan\theta\Delta(\theta).$$

As we proceed along the hyperbola towards infinity ON diminishes, and ultimately vanishes when PN coincides with the asymptote. Also, when this limit is reached, $\sin\omega = k$, and, therefore, $\theta = \pi/2$. Hence, the difference between the asymptote OR and the infinite hyperbolic arc AR is equal to

$$\int_0^{\frac{1}{2}\pi}\frac{ck^2\cos^2\theta\,d\theta}{\Delta(\theta)} = c(E - k'^2K). \qquad (17)$$

Examples.

1. If p, p' are the perpendiculars ON, ON', in Art. 185, show that their product is equal to the product of the semiaxes.

2. Show that the locus of the intersection of the tangents at the points P, P' is the confocal hyperbola

$$\frac{x^2}{a} - \frac{y^2}{b} = a - b,$$

and show that this hyperbola cuts the ellipse in the point Q.

3. Show that a point Q can be found on an hyperbola, such that

$$\text{arc } AQ = \tfrac{1}{2}\{b + \sqrt{(a^2+b^2)} - G\},$$

where G is the difference between the asymptote and the infinite hyperbolic arc measured from the vertex.

187. We now give Dr. Graves's theorem on the arcs of an ellipse. This theorem is the geometrical interpretation of the most general form of the comparison of elliptic integrals of the second kind.

If from any point P, on the outer of two confocal ellipses, tangents PM, PN be drawn to the inner, the sum of the

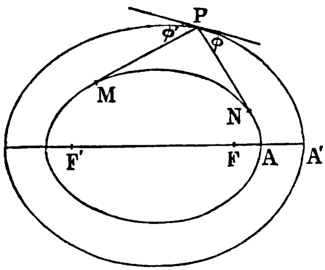

Fig. 31.

tangents diminished by the arc MN between the points of contact is a constant quantity.

To prove this, we first demonstrate the following result. Suppose a line of length r to move between two curves, the elements of whose arcs are ds and $d\sigma$, then

$$dr = \frac{dr}{ds}ds + \frac{dr}{d\sigma}d\sigma,$$

where dr/ds, $dr/d\sigma$ are partial differential coefficients; but by the Differential Calculus we know that

$$\frac{dr}{ds} = \cos\theta, \quad \frac{dr}{d\sigma} = \cos\phi;$$

where θ, ϕ are the angles which the line makes with the tangents to the two curves, respectively.

Now, let the line touch one of the curves; then

$$\cos\theta = 1 \text{ or } -1,$$

and
$$dr = \pm\, ds + \cos\phi\, d\sigma, \tag{18}$$

the sign depending on the direction in which r and s are measured. Hence, applying this result to the case we are considering, and observing that by a property of confocal conics the angles which PM, PN make with the tangent at P are equal, we have

$$dPM = d\,\text{arc}\,AM + \cos\phi\, d\sigma,$$

$$dPN = -\,d\,\text{arc}\,AN - \cos\phi\, d\sigma.$$

Hence

$$d(PM + PN) = d\,\text{arc}\,AM - d\,\text{arc}\,AN = d\,\text{arc}\,MN;$$

whence, by integration, we obtain

$$PM + PN - \text{arc}\,MN = \text{a constant}, \tag{19}$$

which was to be proved.

It may be observed that an exactly similar relation holds in the case of two confocal hyperbolae.

188. If one conic is an ellipse and the other an hyperbola, as in Fig. 32, we have

$$dPM = d\,\text{arc}\,AM + \cos\phi\, d\sigma,$$

$$dPN = -\,d\,\text{arc}\,AN + \cos\phi\, d\sigma;$$

therefore

$$\begin{aligned} d(PM - PN) &= d\,\text{arc}\,AM + d\,\text{arc}\,AN \\ &= d(\text{arc}\,AK + KM) + d(\text{arc}\,AK - \text{arc}\,NK) \\ &= d\,\text{arc}\,KM - d\,\text{arc}\,NK, \end{aligned}$$

where K is the point where the hyperbolae meets the ellipse. Hence, integrating, we get

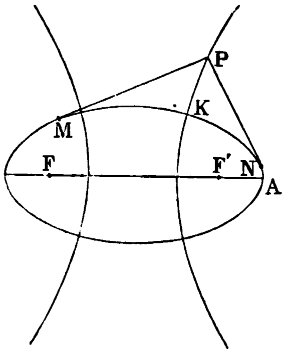

Fig. 32.

$$PM - PN = \text{arc } KM - \text{arc } NK, \tag{20}$$

as the constant vanishes when P coincides with K.

The latter result is due to Professor Mac Cullagh, and it may be added, that both theorems appear to have been obtained independently by Chasles.

The relation (20) still holds if the tangents be drawn from a point on an ellipse to a confocal hyperbola, provided the tangents touch the same branch of the hyperbola, as can be proved in an exactly similar manner.

189. From the theory of elliptic integrals, it follows at once from the preceding results, that if we put

$$u = \int_0^\phi \frac{d\phi}{\sqrt{(1 - e^2 \sin^2\phi)}} = F(\phi),$$

where ϕ has the same value as in Art. 176; then the tangents whose parameters are u_1, u_2 intersect on a confocal ellipse, if

$u_1 - u_2$ is given, and on a confocal hyperbola, if $u_1 + u_2$ is given.

We can hence easily express the quantities $u_1 \pm u_2$ in terms of the principal semiaxes μ, ν of the confocal conics which pass through a point. Taking the vertex A' in Fig. 31 as a particular position of P, we have $u_2 = - u_1$, and, therefore,

$$\operatorname{sn} \frac{u_1 - u_2}{2} = \sin \phi,$$

where ϕ belongs to the point of contact of a tangent drawn from A'. But, putting $y = 0$ in the equation of this tangent, namely,

$$\frac{x}{a} \sin \phi + \frac{y}{b} \cos \phi - 1 = 0,$$

we get $\mu \sin \phi = a$, observing that x is then equal to μ.

Again, in Fig. 32, letting P coincide with K, we get

$$\operatorname{sn} \frac{u_1 + u_2}{2} = \sin \psi,$$

where ψ is the parameter belonging to K. But

$$a \sin \psi = x, \quad \text{and} \quad cx = a\nu;$$

therefore $$c \sin \psi = \nu, \quad \text{where} \quad FF' = 2c.$$

We thus have, finally,

$$\operatorname{sn} \frac{u_1 - u_2}{2} = \frac{a}{\mu}, \quad \operatorname{sn} \frac{u_1 + u_2}{2} = \frac{\nu}{c}. \tag{21}$$

190. In connection with this subject, we may notice that Chasles has given some interesting results connecting circles with arcs whose difference is rectifiable.

Let P, P' be two points on a confocal conic, as in Fig. 31 or Fig. 32; then the four tangents drawn to the given curve from these points are all touched by the same circle; for if R is the intersection of the tangents to the confocal at P, P', it is easily shown that the perpendiculars from R on the four tangents are all equal; in fact, two of the perpendiculars are each equal to $PR \sin \phi$, and the other two to $P'R \sin \phi'$; but the tangents PR, $P'R$ are proportional to the parallel diameters, and $\sin \phi$, $\sin \phi'$ (see Salmon's *Conics*, Art. 189), are inversely proportional to the same quantities. Hence we see that the tangents at the extremities of two arcs of a conic, whose difference is rectifiable, are all touched by the same circle.

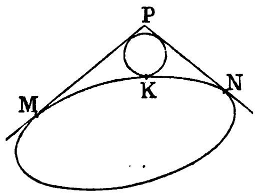

Fig. 33.

If we suppose two of the tangents to coincide, the circle will touch the conic, and we arrive at the following theorem of Chasles. If a circle be described to touch a conic and two of its tangents PM, PN, the point of contact K divides the arc MN into two parts, whose difference is equal to the difference of the tangents PM, PN.

191. We may notice here the application of Landen's transformation in elliptic integrals to the arcs of conics. We have proved already, in Art. 96, that if

$$\lambda^2 = 4k/(1+k)^2, \ \sin(2\phi - \theta) = k \sin \theta;$$

then $\quad k'^2 F_k(\theta) = 2E_k(\theta) - 2(1+k)E_\lambda(\phi) + 2k\sin\theta.$

Hence, in the same case, we have

$$c\tan\theta\Delta(\theta) - cE_k(\theta) + ck'^2F_k(\theta)$$
$$= cE_k(\theta) - 2c(1+k)E_\lambda(\phi) + c\tan\theta\Delta(\theta) + 2ck\sin\theta. \quad (22)$$

Now, the expression on the left-hand side of this equation is, from Art. 186, the value of a hyperbolic arc, and $E_k(\theta)$, $E_\lambda(\phi)$ are each equal to the arcs of certain ellipses; so that we see, from (22), that an arc of a hyperbola can be expressed by the arcs of two ellipses and a portion of a right line.

Putting $\phi = \pi/2$, and, therefore, $\theta = \pi$, the left-hand side of (22) becomes equal to double the difference between the asymptote and the infinite hyperbolic arc, and the right-hand side is then

$$2cE_k - 2c(1+k)E_\lambda.$$

We thus easily find that the difference we have just mentioned is equal to the difference of the quadrants of two ellipses whose semiaxes are $c(1+k)$, $c(1-k)$, and c, ck', respectively.

Examples.

1. Show that the point which divides an elliptic quadrant into parts, whose difference is equal to that of the semiaxes, can be found at once from MacCullagh's theorem.

2. If l denote the sum of the arc and tangent of an ellipse, measured to a point P, show that

$$dl = \sqrt{\left(\frac{\mu^2 - a^2}{\mu^2 - c^2}\right)}\, d\mu \pm \sqrt{\left(\frac{a^2 - \nu^2}{c^2 - \nu^2}\right)};$$

where μ, ν are elliptic co-ordinates—namely, the principal semiaxes of the confocal conics passing through P, and $\mu = a$ is the given curve. Hence obtain the differential equation of the system of involutes of a conic.

3. Show that the sides of a polygon of maximum perimeter inscribed in a conic all touch the same confocal conic.

4. If $n-1$ vertices of a polygon of n sides circumscribed about an ellipse move on confocal ellipses, show that the n^{th} vertex will also move on a confocal ellipse, and that the perimeter of the polygon is constant.

5. If ds is the element of the arc of a curve, show that

$$ds^2 = \left(\frac{\mu^2 - \nu^2}{\mu^2 - c^2}\right) d\mu^2 + \left(\frac{\mu^2 - \nu^2}{c^2 - \nu^2}\right) d\nu^2,$$

where μ, ν are elliptic co-ordinates.

6. If μ, ν are the elliptic co-ordinates of any point on a fixed tangent to the ellipse $\mu = a$, show that

$$\frac{d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = 0.$$

7. With the notation of the preceding example, show, more generally, that if the tangent be variable,

$$\frac{d\omega}{\sqrt{(a^2 - c^2\sin^2\omega)}} = \frac{d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}},$$

where $x\cos\omega + y\sin\omega = \sqrt{(a^2 - c^2\sin^2\omega)}$ is the equation of the tangent.

8. If u has the same meaning as in Art. 189, show that if four tangents are touched by the same circle, $\Sigma_1^4(\pm u_r) = 4mK$, where m is any integer.

Also show that this result can be deduced from the particular case of Abel's theorem, given in Art. 100, by expressing that a tangent of the conic touches a circle, and putting, then, $\tan\frac{1}{2}\phi = x$.

192. We now proceed to show how we may obtain the arc of a curve, which is known to be the inverse of a given curve with regard to some point.

Taking the point as origin, for the inverse curve, we put k^2/r instead of r, and leave θ unaltered. Hence, if ds' is the element of the arc of the inverse curve, we have

$$ds'^2 = dr'^2 + r'^2 d\theta'^2 = \frac{k^4}{r^4}\,dr^2 + \frac{k^4}{r^2}\,d\theta^2$$

$$= \frac{k^4}{r^4}\,(dr^2 + r^2 d\theta^2) = \frac{k^4}{r^4}\,ds^2.$$

Hence
$$ds' = \frac{k^2}{r^2}\,ds,$$

and
$$s' = \int \frac{k^2\,ds}{r^2}. \qquad (23)$$

If we transform to rectangular co-ordinates, we get
$$s' = \int \frac{k^2\,ds}{(x-a)^2 + (y-\beta)^2}, \qquad (24)$$

where a, β are the co-ordinates of the origin of inversion.

193. As an example, let us consider the inverse of the parabola $y^2 - 4mx = 0$, with regard to the point a, β.

Now, for any point xy on the parabola, we may put $x = m\mu^2$, $y = 2m\mu$, where μ is a variable parameter. We have, then,
$$ds^2 = dx^2 + dy^2 = 4m^2(1+\mu^2)\,d\mu^2.$$

Hence we get, from (24),
$$s' = \int \frac{2mk^2\sqrt{(1+\mu^2)}\,d\mu}{(a - m\mu^2)^2 + (\beta - 2m\mu)^2}. \qquad (25)$$

We see thus that the arc of the curve we are considering is always expressible by logarithmic or circular functions.

It may be of interest to effect the actual integration in the case in which $\beta = 0$, namely, when the origin of inversion lies on the axis of the parabola. We have, then, if we put $\mu = \tan\omega$,
$$s' = \int \frac{2mk^2\cos\omega\,d\omega}{(a\cos^2\omega - m\sin^2\omega)^2 + 4m^2\sin^2\omega\cos^2\omega}$$
$$= \int \frac{2mk^2\,d\sin\omega}{(a-m)(a+3m)\sin^4\omega - 2(a-m)(a+2m)\sin^2\omega + a^2}$$
$$= \int \frac{k^2\,d\sin\omega}{\sqrt{(m^2 - ma)}}\left\{\frac{p^2}{1 - p^2\sin^2\omega} - \frac{q^2}{1 - q^2\sin^2\omega}\right\}.$$

where
$$p^2 = \frac{m - a + 2\sqrt{(m^2 - ma)}}{\{\sqrt{(m - a)} + \sqrt{m}\}^2};$$
$$q^2 = \frac{m - a - 2\sqrt{(m^2 - ma)}}{\{\sqrt{(m - a)} - \sqrt{m}\}^2}.$$

In order that p^2 and q^2 should be both positive and real, it is necessary that
$$a + 3m > 0, \quad m > a.$$

If this be the case, we have, integrating and measuring the arc from $\omega = 0$,
$$s' = \frac{k^2}{2\sqrt{(m^2 - ma)}}\left\{p \log\left(\frac{1 + p\sin\omega}{1 - p\sin\omega}\right) - q\log\left(\frac{1 + q\sin\omega}{1 - q\sin\omega}\right)\right\}. \tag{26}$$

In particular, it may be noticed, that if $a = -3m$, then $q = 0$, and
$$s' = \frac{k^2}{3m\sqrt{2}} \log\left(\frac{3 + 2\sqrt{2}\sin\omega}{3 - 2\sqrt{2}\sin\omega}\right). \tag{27}$$

If a does not lie within the limits given above, the arc will be expressible in terms of a logarithm and a circular function.

194. Proceeding to the case of the inverse of an ellipse, with regard to the centre, we have, with the substitution of Art. 176,
$$s' = \int \frac{k^2\sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}}{a^2\sin^2\phi + b^2\cos^2\phi} d\phi$$
$$= \int \left\{\frac{k^2(a^2 + b^2)}{a^2\sin^2\phi + b^2\cos^2\phi} - k^2\right\} \frac{d\phi}{\sqrt{(a^2\cos^2\phi + b^2\sin^2\phi)}}$$
$$= \frac{k^2(a^2 + b^2)}{ab^2}\Pi(n, \phi) - \frac{k^2}{a}F(\phi), \tag{28}$$

where $n = (a^2 - b^2)/b^2$, and the modulus is equal to the eccentricity of the ellipse. Similarly, for the inverse of the hyperbola, we have, with the notation of Art. 177,

$$s' = \int \frac{k^2 \sqrt{(b^2 + a^2 \cos^2 \phi)}}{a^2 + b^2 \cos^2 \phi} d\phi$$

$$= \int \left\{ \frac{k^2 a^2}{b^2} - \frac{k^2 (a^4 - b^4)}{b^2 (a^2 + b^2 \cos^2 \phi)} \right\} \frac{d\phi}{\sqrt{(b^2 + a^2 \cos^2 \phi)}}$$

$$= \frac{k^2 a^2}{b^2 \sqrt{(a^2 + b^2)}} F(\phi) - \frac{k^2 (a^2 - b^2)}{b^2 \sqrt{(a^2 + b^2)}} \Pi(n, \phi), \qquad (29)$$

where $\qquad n = - b^2/(a^2 + b^2), \quad k = a/\sqrt{(a^2 + b^2)},$

and the arc is measured from the origin. More generally, we may observe that the arc of the inverse of a conic section with respect to any point α, β, can be expressed by elliptic integrals. Thus, in the case of the ellipse, we have, from (24),

$$s' = \int \frac{k^2 \sqrt{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)} d\phi}{(\alpha - a \sin\phi)^2 + (\beta - b \cos \phi)^2}.$$

But this integral, if we put

$$\cos \phi = \frac{1 - t^2}{1 + t^2}, \quad \sin \phi = \frac{2t}{1 + t^2},$$

assumes a form which is shown in Chapter V. to be reducible to elliptic integrals.

195. If a curve be given as the envelope of the line $x \cos \omega + y \sin \omega - p = 0$; then, since for the given curve

$$\frac{ds}{d\omega} = p + \frac{d^2 p}{d\omega^2}, \quad r^2 = p^2 + \left(\frac{dp}{d\omega}\right)^2,$$

the arc s' of the inverse curve is given by the equation

$$s' = \int \frac{k^2\left(p + \frac{d^2p}{d\omega^2}\right)d\omega}{p^2 + \left(\frac{dp}{d\omega}\right)^2}. \tag{30}$$

This expression, therefore, gives the arc of the envelope of the circle

$$p\,(x^2 + y^2) - k^2\,(x \cos\omega + y \sin\omega) = 0,$$

as we see by inverting the equation of the line.

As an example of the use of this formula, let us consider the curve

$$(cx)^{\frac{2}{3}} + (cy)^{\frac{2}{3}} = (x^2 + y^2)^{\frac{2}{3}}.$$

This curve is the envelope of the circle

$$\sin 2\omega\,(x^2 + y^2) - 2c\,(x \cos\omega + y \sin\omega) = 0;$$

so that we may take

$$k = c, \quad p = c \sin\omega \cos\omega.$$

We have, then,

$$p + \frac{d^2p}{d\omega^2} = -3c \sin\omega \cos\omega, \quad p^2 + \left(\frac{dp}{d\omega}\right)^2 = \frac{c^2}{4}\,(\sin^2 2\omega + 4\cos^2 2\omega);$$

therefore,

$$\begin{aligned} s' &= \int \frac{-12c \sin\omega \cos\omega\, d\omega}{\sin^2 2\omega + 4\cos^2 2\omega} \\ &= \int \frac{3c\,d\,(\cos 2\omega)}{1 + 3\cos^2 2\omega} \\ &= c\sqrt{3}\left\{\frac{\pi}{3} - \tan^{-1}(\sqrt{3}\cos 2\omega)\right., \end{aligned}$$

if the arc be measured from the point for which $\omega = 0$, namely,

$$x = c, \quad y = 0.$$

EXAMPLES.

1. A curve is defined by the equations

$$x = \frac{m \sin^2\theta}{9 + \tan^2\theta}, \quad y = \frac{m \sin 2\theta}{9 + \tan^2\theta};$$

show that the arc of the curve lying on one side of the axis of x is bisected at the point for which $\sin\theta = 3/4$.

2. Show that the arc of the inverse of a parabola, with regard to a point on itself, can be expressed by the integral

$$\frac{2k^2}{m} \int \frac{\sqrt{(1+\mu^2)}\, d\mu}{(\mu - \theta)^2 \{(\mu + \theta)^2 + 4m^2\}}.$$

3. Show that the arc of the central inverse of a conic can be expressed by the integral

$$\int \frac{a^2 b^2 k^2}{a^4 \cos^2\omega + b^4 \sin^2\omega} \frac{d\omega}{\sqrt{(a^2\cos^2\omega \pm b^2 \sin^2\omega)}}.$$

4. If s is the arc of the inverse of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

with regard to the point $(a^2 + b^2)/c,\ 0$, show that

$$s = \int \left\{ \frac{2a(a^2+b^2)h}{c^3(h^2 - c^2\sin^2\phi)} + \frac{2a(a^2+b^2)\sin\phi}{c(h^2 - c^2\sin^2\phi)} - 1 \right\} \times \frac{k^2\, d\phi}{\sqrt{(a^2 - c^2\sin^2\phi)}},$$

where $$a^2 - b^2 = c^2, \quad h = a(a^2 + 3b^2)/c^2.$$

5. If s is the arc of the envelope of the circle

$$\cos n\omega (x^2 + y^2) - b(x\cos\omega + y \sin\omega) = 0,$$

show that

$$s = \frac{b(1-n^2)}{n} \int \frac{dz}{1 - (1-n^2)z^2},$$

where $z = \sin n\omega$.

196. The inverse curves of a conic can also be rectified by considering them as pedals; for we know that the inverse of a conic with regard to a point is the pedal of the reciprocal conic with regard to that point. Now, if s is the arc of the pedal, we have (see fig. 22),

$$\left(\frac{ds}{d\omega}\right)^2 = p^2 + \left(\frac{dp}{d\omega}\right)^2 = OS^2 + OR^2 = OP^2 = r^2;$$

therefore $$s = \int r d\omega. \tag{31}$$

Applying this result to the case of the ellipse, we get

$$s = \int \frac{ab\sqrt{\{(\alpha - a\sin\phi)^2 + (\beta - b\cos\phi)^2\}}\, d\phi}{a^2\cos^2\phi + b^2\sin^2\phi}, \tag{32}$$

since $$r^2 = (\alpha - a\sin\phi)^2 + (\beta - b\cos\phi)^2,$$

$$d\omega = \frac{abd\phi}{a^2\cos^2\phi + b^2\sin^2\phi}.$$

197. Of more general curves of the third and fourth degree, the only ones whose rectification has been considered are the circular cubics and bicircular quartics, namely, the curves whose mode of generation has been already given in Art. 150. An Italian mathematician, Genocchi, first showed that the arc of the curve called the Cartesian oval can be expressed in terms of arcs of three ellipses; and, since then, Dr. Casey has demonstrated, by geometrical considerations, that the arc of the general bicircular quartic can be determined linearly in terms of four expressions, each of which is reducible to elliptic integrals. We propose to give here a slight account of the latter result. If r, θ are polar coordinates, we have

$$ds^2 = dr^2 + r^2 d\theta^2;$$

but from the mode of generation given in Art. 150,

$$r = p \pm \sqrt{(p^2 - k^2)}, \quad \theta = \omega.$$

Hence, if ds, ds' refer to the points Q, P, respectively, we get

$$ds = \{p + \sqrt{(p^2 - k^2)}\}\sqrt{\left\{\frac{dp^2}{p^2 - k^2} + d\omega^2\right\}},$$

$$ds' = \{p - \sqrt{(p^2 - k^2)}\}\sqrt{\left\{\frac{dp^2}{p^2 - k^2} + d\omega^2\right\}}.$$

We thus have

$$d(s + s') = \frac{2p}{\sqrt{(p^2 - k^2)}}\sqrt{\{dp^2 + (p^2 - k^2)\,d\omega^2\}},$$

$$d(s - s') = 2\sqrt{\{dp^2 + (p^2 - k^2)\,d\omega^2\}};$$

whence we obtain

$$s + s' = 2\int \frac{p\,d\omega}{\sqrt{(p^2 - k^2)}}\sqrt{\left\{\left(\frac{dp}{d\omega}\right)^2 + p^2 - k^2\right\}}, \qquad (33)$$

$$s - s' = 2\int \sqrt{\left\{\left(\frac{dp}{d\omega}\right)^2 + p^2 - k^2\right\}}\,d\omega. \qquad (34)$$

The latter formula may be written

$$s - s' = 2\int \sqrt{(r^2 - k^2)}\,d\omega, \qquad (35)$$

where $OR = r$; and being applied then to the case under consideration, gives (see Art. 150),

$$s - s' = 2\int \frac{ab\sqrt{\{(\alpha - a\cos\phi)^2 + (\beta - b\sin\phi)^2 - k^2\}}\,d\phi}{a^2\sin^2\phi + b^2\cos^2\phi}, \qquad (36)$$

which, by putting

$$\cos\phi = \frac{1 - t^2}{1 + t^2}, \quad \sin\phi = \frac{2t}{1 + t^2},$$

assumes a form reducible to elliptic integrals.

It is by means of this result, and a geometrical property of the curve, that Dr. Casey demonstrates his theorem. This property is, that the curve can be generated in the manner described in Art. 150 in four distinct ways, the four points such as O being the centres of four mutually orthogonal circles, with regard to each of which the curve is its own inverse. Also the following geometrical theorem is required: If any figure be inverted successively with regard to four mutually orthogonal circles, the fourth inversion will coincide with the original figure.

Suppose now that P is a point on the curve, P_1 its inverse with regard to the first circle, and P_2 the inverse of P_1 with regard to the second circle, &c.; then, by the second of the propositions stated above, the point P_4 coincides with P. Hence, if ds, ds_1, &c., correspond to the points P, P_1, &c., respectively, we have, from (36), in which we write the integral in the form $2\int N d\phi$,

$$\left.\begin{aligned} s - s_1 &= 2\int N_1 d\phi_1, \\ s_1 - s_2 &= 2\int N_2 d\phi_2, \\ s_2 - s_3 &= 2\int N_3 d\phi_3, \\ s_3 + s &= 2\int N_4 d\phi_4, \end{aligned}\right\} \qquad (37)$$

where it is to be observed that the last arc, s_4 or s, is taken with a negative sign, as, if this were not the case, we should have, by addition, a relation of the form $\Sigma N d\phi = 0$, which, it is easily shown, could not exist.

Hence, adding these results together, we eliminate s_1, s_2, s_3, and find

$$s = \int N_1 d\phi_1 + \int N_2 d\phi_2 + \int N_3 d\phi_3 + \int N_4 d\phi_4, \qquad (38)$$

which is Dr. Casey's expression for the arc of a bicircular quartic.

198. If the conic in Fig. 18 becomes a circle, the locus of P, Q is a Cartesian oval, and there are then only three distinct ways of generating the curve. In this case the third inversion of a point P is the reflexion of P with regard to the axis of symmetry, so that $ds_3 = -ds$. We find thus that the arc is equal to the sum of three expressions of the form $\int N d\phi$, where N now becomes

$$\sqrt{(a^2 + \alpha^2 - k^2 - 2a\alpha\cos\phi)} \quad \text{by putting} \quad b = a,\ \beta = 0.$$

But the latter integral is equal to an arc of an ellipse, the squares of whose semiaxes are

$$4\{(a \pm \alpha)^2 - k^2\}.$$

We see thus that an arc of the Cartesian oval can be expressed linearly in terms of the arcs of three ellipses, as was stated above.

It may be observed that the axes of the ellipses can be readily expressed in terms of lines connected with the curves.

Putting $\qquad OP = \rho, \quad p = a - \alpha\cos\omega,$

we have (see Art. 150),

$$\rho^2 - 2\rho(a - \alpha\cos\omega) + k^2 = 0;$$

hence, if we put $\omega = \pi$ or 0, we get, to determine the points A, B, C, D, where the axis of symmetry meets the curve,

$$\rho^2 - 2\rho(a \pm \alpha) + k^2 = 0,$$

from which we find AB and CD, say, equal to

$$2\sqrt{\{(a + \alpha)^2 - k^2\}}, \quad 2\sqrt{\{(a - \alpha)^2 - k^2\}},$$

respectively. It thus appears that the semiaxes of one of the ellipses are the lengths AB, CD. Hence, by symmetry, we infer that the semiaxes of the three ellipses are the lines AB, CD; AC, BD; AD, BC, respectively.

199. We now consider in particular the bicircular quartic with a centre, that is, the curve generated in the case when the point O in Fig. 18 coincides with the centre of the conic. Since in this case

$$p^2 = a^2 \cos^2\omega + b^2 \cos^2\omega,$$

the polar equation of the curve is

$$r = \sqrt{(a^2 \cos^2\omega + b^2 \sin^2\omega)} \pm \sqrt{(a'^2 \cos^2\omega + b'^2 \sin^2\omega)}, \quad (39)$$

where we have put a'^2, b'^2 for $a^2 - k^2$, $b^2 - k^2$, respectively. It thus appears that the quartic could be generated in exactly the same way from the confocal conic

$$\frac{x^2}{a^2 - k^2} + \frac{y^2}{b^2 - k^2} - 1 = 0,$$

the constant k^2 being replaced by $-k^2$. In this case (36) gives

$$s - s' = 2ab \int \frac{\sqrt{(a'^2 \cos^2\phi + b'^2 \sin^2\phi)}\, d\phi}{a^2 \sin^2\phi + b^2 \cos^2\phi}$$

$$= 2ab \int \left\{ \frac{a^2 + b^2 - k^2}{a^2 \sin^2\phi + b^2 \cos^2\phi} - 1 \right\} \frac{d\phi}{\sqrt{(a'^2 \cos^2\phi + b'^2 \sin^2\phi)}}$$

$$= \frac{2a(a^2 + b^2 - k^2)}{b\sqrt{(a^2 - k^2)}} \Pi(n, \phi) - \frac{2ab}{\sqrt{(a^2 - k^2)}} F(\phi), \quad (40)$$

where the square of the modulus is $(a^2 - b^2) / (a^2 - k^2)$,

$$n = (a^2 - b^2) / b^2,$$

and k is supposed to be less than b.

Again, by means of (33), or directly from (35) and the second mode of generation, we find

$$s + s' = \frac{2(a^2 + b^2 - k^2)\sqrt{(b^2 - k^2)}}{a\sqrt{(a^2 - k^2)}}\Pi(n', \theta)$$

$$- \frac{2}{a}\sqrt{\{(a^2 - k^2)(b^2 - k^2)\}}F(\theta), \qquad (41)$$

where the square of the modulus is $(a^2 - b^2)/a^2$,

$$n' = (a^2 - b^2)/(b^2 - k^2),$$

and θ is the eccentric angle of the point on the second conic,

namely, $$\tan^2\theta = \left(\frac{b^2 - k^2}{a^2 - k^2}\right)\tan^2\omega.$$

In the case considered, that is, when $k < b$, it is easy to see that the curve consists of two ovals, one wholly within the other; but if k is intermediate between a and b, the two ovals are external to each other, and the conic is a hyperbola. In the latter case, of course, the expression for the sum of the arcs is reduced to the standard forms in a different way. The transformation, in fact, then is

$$\tan\omega = \sqrt{\left(\frac{a^2 - k^2}{k^2 - b^2}\right)}\sin\theta; \qquad (42)$$

so that we get

$$s + s' = 2\sqrt{\{(a^2 - k^2)(k^2 - b^2)\}}\int\frac{\sqrt{(a^2 - b^2\sin^2\theta)}\,d\theta}{k^2 - b^2 + (a^2 - k^2)\sin^2\theta}, \qquad (43)$$

which can be readily shown to depend upon elliptic integrals whose modulus is b/a, and parameter $(a^2 - k^2)/(k^2 - b^2)$.

If both the generating conics are hyperbolæ, we have to

make use of a transformation such as (42) in the expression for the difference of the arcs also.

200. An interesting particular case of the curve considered in the preceding Article is the Cassinian oval or ovals, some properties of which we have already given in Arts. 153, 154.

If we clear (39) of radicals, we get

$$r^4 - 2r^2\{(2a^2 - k^2)\cos^2\omega + (2b^2 - k^2)\sin^2\omega\} + k^4 = 0;$$

but in order that this should represent a Cassinian, we must have the coefficient of r^2 proportional to $\cos 2\omega$; hence we get $k^2 = a^2 + b^2$. But when this relation is satisfied, it is easy to see that if the curve is real, the generating conic must be a hyperbola. In fact the two generating conics are then the hyperbolæ

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0, \quad \frac{x^2}{b^2} - \frac{y^2}{a^2} - 1 = 0,$$

where we have changed the sign of b^2.

We find then from (33) and (34)

$$s = \int \frac{ab\,d\omega}{\sqrt{(a^2\cos^2\omega - b^2\sin^2\omega)}} \pm \int \frac{ab\,d\omega}{\sqrt{(b^2\cos^2\omega - a^2\sin^2\omega)}}$$

$$= \frac{ab}{\sqrt{(a^2 + b^2)}}\{F_k(\phi) \pm F_{k'}(\psi)\}, \qquad (44)$$

where $\qquad a\sin\phi = b\sin\psi = \sqrt{(a^2 + b^2)}\sin\omega,$

the modulus is $\qquad a/\sqrt{(a^2 + b^2)},$

and the arcs of the oval are measured from the points for which $\omega = 0$. Since the curve may be written

$$(x^2 + y^2 + a^2 - b^2)^2 - 4(a^2x^2 - b^2y^2) = 0,$$

we see that b must be greater than a, if

$$a^2x^2 - b^2y^2 = 0$$

represent the real tangents drawn from the origin. Hence, if this be the case, we have for these tangents

$$\sin\omega = \pm a/\sqrt{(a^2 + b^2)};$$

and, therefore, $$\phi = \pm \pi/2.$$

But we have

$$s + s' = \frac{2ab}{\sqrt{(a^2 + b^2)}} F_k(\phi);$$

so that we find that the entire perimeter of either of the ovals is

$$\frac{4abK}{\sqrt{(a^2 + b^2)}}.$$

Also, it may be observed that the points of contact of the tangents from the origin divide the oval into two parts, the difference of whose lengths is equal to

$$\frac{4abK'}{\sqrt{(a^2 + b^2)}}.$$

201. When the Cassinian is generated in the manner considered in the preceding Article, it always consists of two ovals exterior to each other; but we have seen already in Art. 153 that the curve may consist of a single oval enclosing the origin, in which case its polar equation takes the form

$$r^4 - 2c^2r^2\cos 2\theta - k^4 = 0.$$

The mode of generation by means of the two hyperbolæ

becomes then imaginary, and the rectification of the curve must be effected in a different manner, as follows:—

We have

$$r^2 = c^2 \cos 2\theta + \sqrt{(k^4 + c^4 \cos^2 2\theta)},$$

which, by putting

$$k^4 + c^4 \cos^2 2\theta = k^4 z^2,$$

gives

$$r^2 = k^2 \{z + \sqrt{(z^2 - 1)}\},$$

and, therefore,

$$r = \frac{k}{\sqrt{2}} \{\sqrt{(z + 1)} + \sqrt{(z - 1)}\}.$$

Now we have

$$\frac{dr}{rd\theta} = -\frac{c^2 \sin 2\theta}{\sqrt{(k^4 + c^4 \cos^2 2\theta)}};$$

hence

$$\frac{ds}{d\theta} = r\sqrt{\left\{1 + \left(\frac{dr}{rd\theta}\right)^2\right\}} = \frac{r\sqrt{(c^4 + k^4)}}{\sqrt{(k^4 + c^4 \cos^2 2\theta)}};$$

but

$$2d\theta = -\frac{k^2 z dz}{\sqrt{\{(z^2 - 1)(c^4 + k^4 - k^4 z^2)\}}};$$

so that we get

$$ds = -\frac{k\sqrt{(c^4 + k^4)}}{2\sqrt{2}} \left\{\frac{dz}{\sqrt{\{(z-1)(c^4 + k^4 - k^4 z^2)\}}} + \frac{dz}{\sqrt{\{(z+1)(c^4 + k^4 - k^4 z^2)\}}}\right\},$$

which, by putting $c^4 + k^4 = k^4 \sec^2 2a$, $z = \sec 2a \cos 2\phi$, gives

$$s = \frac{k}{\sqrt{(2 \cos 2a)}} \left\{\int \frac{d\phi}{\sqrt{(\cos 2\phi - \cos 2a)}} + \int \frac{d\phi}{\sqrt{(\cos 2\phi + \cos 2a)}}\right\}$$

$$= \frac{k}{2\sqrt{\cos 2a}} \{F_\lambda(\psi_1) + F_{\lambda'}(\psi_2)\}, \qquad (45)$$

where $\sin\phi = \sin\alpha \sin\psi_1 = \cos\alpha \sin\psi_2$,

and the moduli λ, λ' are $\sin\alpha$, $\cos\alpha$, respectively.

If we considered a radius vector at right angles to that corresponding to the angle θ, we should have

$$r'^2 = -c^2 \cos 2\theta + \sqrt{(k^4 + c^4 \cos^2 2\theta)},$$

and, proceeding then as above, we should find

$$s' = \frac{k}{2\sqrt{\cos 2\alpha}} \{F_\lambda(\psi_1) - F_{\lambda'}(\psi_2)\}. \tag{46}$$

Hence we have

$$s + s' = \frac{k}{\sqrt{\cos 2\alpha}} F_\lambda(\psi_1),$$

$$s - s' = \frac{k}{\sqrt{\cos 2\alpha}} F_{\lambda'}(\psi_2).$$

Taking $\theta = \pi/4$, we have $\phi = \alpha$, and, therefore, $\psi_1 = \pi/2$, so that from the first of these equations we find that the whole perimeter of the curve is

$$\frac{4kK}{\sqrt{\cos 2\alpha}}.$$

The preceding results on the rectification of the Cassinian were first given by Serret (*Journal de Mathématiques*, t. viii., 1843, p. 145).

Examples.

1. If $(a^2 - b^2)(a^2 - k^2) = b^4$, where $2b^2 > a^2$, show that the difference of two arcs of the quartic $(x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2) = 0$ can be expressed by an elliptic integral of the first kind and a circular function.

2. Show that the elliptic integrals of the third kind involved in the expression for the arc of the quartic

$$(x^2 + y^2 + k^2)^2 - 4(a^2x^2 + b^2y^2) = 0$$

have both their parameters circular or logarithmic according as

$$a^2 + b^2 \text{ is } > \text{ or } < k^2.$$

3. If $d\sigma$ is an element of the arc of the quartic

$$r^4 - 2r^2(\alpha\cos^2\theta + \beta\sin^2\theta) + k^4 = 0,$$

show that

$$d\sigma = \sqrt{\left\{\frac{(\alpha^2 - k^4)\cos^2\theta + (\beta^2 - k^4)\sin^2\theta}{2(\alpha\cos^2\theta + \beta\sin^2\theta + k^2)}\right\}}\,d\theta$$

$$\pm\sqrt{\left\{\frac{(\alpha^2 - k^4)\cos^2\theta + (\beta^2 - k^4)\sin^2\theta}{2(\alpha\cos^2\theta + \beta\sin^2\theta - k^2)}\right\}}\,d\theta.$$

4. From the preceding example show that the expression for the element $d\sigma$ of the arc of the quartic

$$r^4 - 2r^2(\alpha\cos^2\theta + \beta\sin^2\theta) - k^4 = 0$$

may be written

$$d\sigma = \sqrt{\left\{\frac{m\cos 2\phi + n\sin 2\phi}{(\alpha\sin 2\phi - k^2\cos 2\phi)(k^2\cos 2\phi - \beta\sin 2\phi)}\right\}}\,\frac{d\phi}{\sin\phi},$$

where $m = \frac{1}{2}k^4(\alpha + \beta)$, $n = \frac{1}{2}k^2(k^4 - \alpha\beta)$, $\alpha\cos^2\theta + \beta\sin^2\theta = k^2\cot 2\phi$.

It may be observed that the arc of the curve in this case cannot be expressed by real elliptic integrals.

5. If a bicircular quartic have an axis of symmetry, show that the arc of the curve can be found as the sum of three expressions, each of which can be evaluated by means of elliptic integrals.

6. If the circle $x^2 + y^2 - 2\alpha x - 2\beta y - (a^2 - b^2) = 0$ have its centre on a conic confocal with the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

where $b > a$, show that it cuts off from the Cassinian

$$(x^2 + y^2)^2 - 2(a^2 + b^2)(x^2 - y^2) + (a^2 - b^2)^2 = 0$$

arcs whose sum or difference is constant.

202. It has been a matter of interest to mathematicians to discover algebraic curves whose arcs can be multiplied or divided algebraically like those of the circle. This can evidently be done, if the arc of the curve is expressible as a circular arc, a logarithm, or an elliptic integral of the first

kind, as the multiplication and division of these functions depend upon algebraic equations. The problem is, therefore, to find curves whose arcs can be represented in the manner just mentioned.

In the case in which the arc is represented by a circular function, Euler found a series of curves satisfying this condition. These curves we investigate here as follows:—Since the arc of the circle is proportional to the angle through which the central radius vector turns, it is hereby suggested to find a curve whose arc varies as the angle which the radius vector makes with the tangent. Let this angle be ϕ; then we take $s = b\phi$. Now

$$dr = \cos\phi\, ds;$$

therefore $$dr = b\cos\phi\, d\phi,$$

whence, by integration,

$$r = a + b\sin\phi.$$

Again,

$$d\theta = \frac{dr}{r}\tan\phi = \frac{b\sin\phi\, d\phi}{a + b\sin\phi},$$

whence, by integration, we get

$$\theta = \phi - \frac{a}{\sqrt{(a^2 - b^2)}}\sin^{-1}\left\{\frac{b + a\sin\phi}{a + b\sin\phi}\right\}$$

(see Ex. 7, p. 28).

Hence, if we put $m^2 b^2 = (m^2 - 1)a^2$, we get

$$\sin\left(\frac{\phi - \theta}{m}\right) = \frac{b + a\sin\phi}{a + b\sin\phi} = \frac{\sqrt{(m^2 - 1)} + m\sin\phi}{m + \sqrt{(m^2 - 1)}\sin\phi}; \qquad (47)$$

but as ϕ is expressed in terms of r by means of the equation

$$r = a + b\sin\phi,$$

it follows that the curve will be algebraic, provided m is commensurable. We thus see that, corresponding to the different values of m, we have a series of algebraic curves whose arcs are proportional to the angle which the radius vector makes with the tangent.

203. It may be noticed that we can generate from these curves a yet more general system whose arcs are expressible by a circular function. For we have seen in Art. 184, that the arc of the parallel curve exceeds that of the given one by $\pm k\omega$; so that if s is the arc of the parallel to one of Euler's curves, we have

$$s = b\phi \pm k\omega = b(\phi \pm n\omega),$$

if we take k the constant interval equal to nb, where n is a commensurable number. In the parallel curve it is to be observed that ϕ is not the angle which the radius vector makes with the curve. It may be considered as a parameter connected with the radius vector r by the equation

$$r^2 = k^2 \pm 2k \sin\phi\,(a + b\sin\phi) + (a + b\sin\phi)^2.$$

204. With respect to curves whose arcs can be represented by a pure logarithm, it may be observed that we have already found, in Art. 193, a curve of the fourth order, whose arc is expressible in this manner. We have also seen that the arc of a more general curve of the same kind is equal to an expression of the form $p\log u + q\log v$, which, by taking p/q equal to a commensurable number, m/n, say, becomes proportional to $\log(u^m v^n)$. We thus have an infinite number of such curves corresponding to the numerical values assigned to m and n.

205. Proceeding to the case of the elliptic integral of the first kind, we notice that we have seen already, in Art. 181,

that the arc of the lemniscate is expressible by such an integral, with a modulus equal to $1/\sqrt{2}$. The arcs of this curve are therefore capable of multiplication and division in the same manner as those of the circle. As an example, we find that if the equation of the curve is $r^2 = a^2 \cos 2\theta$, then the semiperimeter of the loop is bisected at a point whose distance from the pole is $a\sqrt{(2^{\frac{1}{2}} - 1)}$.

In this connection, we may notice some theorems which Chasles deduced from the fact that the lemniscate is the inverse of an equilateral hyperbola with regard to its centre. First: to two arcs of an equilateral hyperbola, whose difference is rectifiable, correspond two equal arcs of the lemniscate. Again, if two circles are drawn through the pole to touch a lemniscate at the points A, B, and another circle be drawn to touch these two circles and the curve, the point of contact of the latter circle is the middle point of the arc AB. Furthermore: the four circles drawn through the pole, to touch a lemniscate at the extremities of two equal arcs, are all touched by the same circle.

The preceding results follow from the properties of arcs of conics already proved in Art. 190, combined with the fact that the arc of the lemniscate is proportional to the argument of the elliptic integral of the second kind, which expresses the arc of the inverse equilateral hyperbola.

206. We thus see, that several interesting geometrical theorems concerning arcs of the lemniscate follow from their being equal to elliptic integrals of the first kind. But as the modulus involved in the arc of the lemniscate is particular, namely, $1/\sqrt{2}$, it becomes a matter of interest to inquire whether there are not other curves, whose arcs represent elliptic integrals of the first kind in a more general manner.

We have seen already, in Arts. 200, 201, that the sum or difference of two arcs of the Cassinian oval is in each case represented by a pure elliptic integral of the first kind; and we have thus, by means of arcs of a curve, a complete geometrical representation of such an integral with an arbitrary modulus. But as the integral is not represented by an arc measured from a fixed point, we cannot multiply or divide algebraically any assigned arc of the curve, so that from this point of view the Cassinian oval cannot be considered a solution of the problem.

207. The most general solution of the question under consideration was given by Serret, who discovered a whole series of systems of algebraic curves corresponding to certain determinate numerical values of the modulus. An algebraic curve, whose arc involves an arbitrary modulus, remains thus to be discovered.

We investigate here merely one of Serret's systems. Let ϕ be the angle which the radius vector makes with the curve; then, if we have $r^2 = b^2 + a^2 \sin\phi$, it is easy to see that the arc is expressible by an elliptic integral of the first kind; for

$$2r\,dr = 2r\cos\phi\,ds = a^2\cos\phi\,d\phi;$$

therefore
$$ds = \frac{a^2}{2}\frac{d\phi}{\sqrt{(b^2 + a^2\sin\phi)}},$$

from which we get

$$s = \frac{a^2}{\sqrt{(a^2+b^2)}}F(\psi),$$

where the modulus is

$$a\sqrt{2}/\sqrt{(a^2+b^2)}, \quad \text{and} \quad \phi = 2\psi + \pi/2.$$

We have now to find whether the equation $r^2 = b^2 + a^2 \sin\phi$ can ever represent an algebraic curve. We have

$$d\theta = \frac{dr}{r}\tan\phi = \frac{a^2}{2}\frac{\sin\phi\, d\phi}{b^2 + a^2\sin\phi};$$

therefore
$$2\theta = \phi - \int \frac{b^2\, d\phi}{b^2 + a^2 \sin\phi}$$

$$= \phi - \frac{b^2}{\sqrt{(b^4 - a^4)}}\sin^{-1}\left\{\frac{b^2 + a^2\sin\phi}{a^2 + b^2\sin\phi}\right\}.$$

Hence, if we take $m^2 a^4 = (m^2 - 1) b^4$, where m is a commensurable number, we have

$$\sin\left(\frac{\phi - 2\theta}{m}\right) = \frac{b^2 + a^2\sin\phi}{a^2 + b^2\sin\phi}; \qquad (48)$$

so that we get thus an infinite number of algebraic curves corresponding to the different values assigned to m.

208. Of these curves, Serret has also given the following geometrical mode of generation. With the radius vector OP

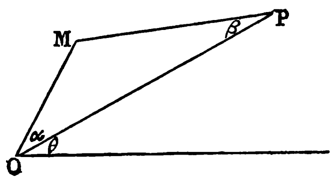

Fig. 34.

as base, let a triangle be constructed whose sides MP, OM are equal to $\sqrt{(n+1)}$, $\sqrt{n}$, respectively; then, if

$$\theta = n\alpha - (n+1)\beta, \qquad (49)$$

where $MOP = \alpha$, $OPM = \beta$, the locus of P is one of Serret's elliptic curves.

To show that the arc of this curve is an elliptic integral, we have, writing for brevity,

$$\Delta = \sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}},$$

$$\cos\alpha = \frac{r^2-1}{2r\sqrt{n}}, \quad \cos\beta = \frac{r^2+1}{2r\sqrt{(n+1)}},$$

$$\sin\alpha = \frac{\Delta}{2r\sqrt{n}}, \quad \sin\beta = \frac{\Delta}{2r\sqrt{(n+1)}},$$

by the ordinary trigonometrical formulae.

We thus find

$$d\alpha = \frac{-(r^2+1)}{\Delta}\frac{dr}{r}, \quad d\beta = \frac{-(r^2-1)}{\Delta}\frac{dr}{r}; \tag{50}$$

hence, from (49),

$$d\theta = n\,d\alpha - (n+1)\,d\beta = \frac{r^2-(2n+1)}{\Delta}\frac{dr}{r},$$

from which we get

$$\frac{ds}{dr} = \sqrt{\left\{1 + \left(\frac{r\,d\theta}{dr}\right)^2\right\}} = \sqrt{\left\{1 + \frac{(r^2-2n-1)^2}{-r^4+2(2n+1)r^2-1}\right\}}$$

$$= \frac{2\sqrt{\{n(n+1)\}}}{\Delta}. \tag{51}$$

We see thus that the arc is expressible as an elliptic integral of the first kind, which, it may be observed, assumes the standard form, if we take the angle α as the variable. We have, from (50) and (51),

$$\frac{ds}{d\alpha} = \frac{ds}{dr}\frac{dr}{d\alpha} = \frac{-2\sqrt{\{n(n+1)\}}}{\Delta}\frac{r\Delta}{r^2+1}$$

$$= \frac{-2\sqrt{\{n(n+1)\}}\,r}{r^2+1};$$

but $$n + 1 - n\sin^2 a = 1 + \frac{(r^2-1)^2}{4r^2} = \frac{(r^2+1)^2}{4r^2};$$

so that we get

$$\frac{ds}{da} = \frac{\sqrt{n}}{\sqrt{(1 - k^2\sin^2 a)}}, \tag{52}$$

where $$k^2 = n/(n+1).$$

209. We might also consider these curves of Serret's from a different point of view, namely, as the inverses of the curves known as epitrochoids. The epitrochoid is the locus of a point rigidly connected with a circle which rolls on the circumference of another; and if the origin be taken at the centre of the fixed circle, the co-ordinates of any point x, y of the locus can be expressed as follows:—

$$x = mb\cos\phi - c\cos m\phi, \quad y = mb\sin\phi - c\sin m\phi. \tag{53}$$

Hence, for the arc of the epitrochoid, we have

$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2$$

$$= m^2(b\sin\phi - c\sin m\phi)^2 + m^2(b\cos\phi - c\cos m\phi)^2$$

$$= m^2(b^2 + c^2 - 2bc\cos\psi), \tag{54}$$

where $$\psi = (m-1)\phi.$$

That is, $$ds = \frac{m}{m-1}\sqrt{(b^2 + c^2 - 2bc\cos\psi)}\,d\psi,$$

from which, it may be observed, we see that the length of the arc is equal to that of an ellipse.

Now we have, from (53),

$$r^2 = m^2 b^2 + c^2 - 2mbc \cos\psi,$$

so that for the arc s' of the inverse we get, from (23),

$$ds' = \frac{k^2}{r^2}\,ds = \frac{mk^2}{m-1}\,\frac{\sqrt{(b^2+c^2-2bc\cos\psi)}\,d\psi}{m^2b^2+c^2-2mbc\cos\psi}$$

$$= \frac{k^2}{m-1}\left\{1 + \frac{(m-1)(c^2-mb^2)}{m^2b^2+c^2-2mbc\cos\psi}\right\}$$

$$\times \frac{d\psi}{\sqrt{(b^2+c^2-2bc\cos\psi)}}.$$

Hence, if we put $\psi = \pi - 2\theta$, we obtain

$$s' = \frac{k^2}{(m-1)(b+c)}\,F(\theta) + \frac{k^2(c^2-mb^2)}{(c+mb)^2(b+c)}\,\Pi(n,\theta), \quad (55)$$

where $$n = -4mbc/(c+mb)^2,$$

and the square of the modulus is $4bc/(b+c)^2$.

We thus see that the arc of the inverse curve represents an elliptic integral of the third kind with an arbitrary modulus. The parameter, however, must be connected with the modulus by some numerical relation if the curve is to be algebraic; for, if λ is the modulus, we have

$$n = -\frac{4m\lambda^2}{\{m+1+(m-1)\sqrt{(1-\lambda^2)}\}^2},$$

and if the curve is algebraic, m must be a commensurable quantity.

Suppose we have now $c^2 = mb^2$, then the coefficient of $\Pi(n, \theta)$ vanishes, and we get

$$s' = \frac{k^2}{b(m-1)(\sqrt{m}+1)} F_\lambda(\theta), \tag{56}$$

where $$\lambda^2 = 4\sqrt{m}/(1+\sqrt{m})^2,$$

so that the length of the arc is then expressible by an elliptic integral of the first kind.

We can easily verify that in this case the curves are those of Serret. From (53) we get

$$\frac{xdy}{d\phi} - \frac{ydx}{d\phi} = m^2b^2 + mc^2 - m(m+1)bc\cos\psi$$

$$= 2m^2b^2 - 2m^{\frac{3}{2}}(m+1)b^2\cos\psi.$$

But $$\frac{xdy}{d\phi} - \frac{ydx}{d\phi} = \frac{pds}{d\phi},$$

and $$\frac{ds}{d\phi} = mb\sqrt{\{m+1-2\sqrt{m}\cos\psi\}};$$

also $$r^2 = m^2b^2 + c^2 - 2mbc\cos\psi$$

$$= mb^2(m+1-2\sqrt{m}\cos\psi);$$

so that $$\frac{ds}{d\phi} = r\sqrt{m}.$$

Hence we have

$$pr\sqrt{m} = 2m^2b^2 - (m+1)\{m(m+1)b^2 - r^2\}$$

$$= (m+1)r^2 - m(m^2+1)b^2.$$

But putting $p = r\sin\chi$, and then substituting k^2/r for r,

and leaving χ unaltered for the inverse curve, we get a relation of the same form as that which we assumed in Art. 207 for Serret's curves, namely,

$$r^2 = b^2 + a^2 \sin \chi.$$

210. We now mention two very simple curves whose arcs are expressible by elliptic integrals of the first kind with numerical moduli. If the arc of the curve

$$r^m = a^m \cos m\theta$$

be expressed in terms of r, we have

$$ds = \frac{a^m dr}{\sqrt{(a^{2m} - r^{2m})}}.$$

Hence, taking $m = 3$, we see that the arc of the curve

$$r^3 = a^3 \cos 3\theta$$

is equal to
$$\frac{a^2}{2}\int \frac{dz}{\sqrt{(z^3 - 1)}},$$

where $r^2 = a^2/z$; but this is an integral of the first kind whose reduction to the standard form is given in Ex. 2, p. 133. Again, let $m = 3/2$, and we have

$$s = \int \frac{a^{\frac{3}{2}} dr}{\sqrt{(a^3 - r^3)}},$$

which is also an integral of the first kind.

211. We now give an interesting theorem of Bernouilli's with respect to the rectification of a certain locus. Let us consider any number n of points on a curve or on different curves at which the tangents are all parallel, and let us take

the locus of the centre of mean position of these points for any system of multiples; then the arc of the locus measured from a properly selected point is equal in length to the mean value for the system of multiples of the arcs described by the points.

To prove this, let $a_1, a_2, \ldots a_n$ be the system of multiples; then if $x_1, y_1, \ldots x_n, y_n$ are the co-ordinates of the points on the curves, and x, y are those of the corresponding point of the locus, we have

$$x\Sigma_1^n a_r = \Sigma_1^n a_r x_r, \quad y\Sigma_1^n a_r = \Sigma_1^n a_r y_r;$$

therefore
$$\frac{dx}{ds} ds\Sigma_1^n a_r = \Sigma_1^n a_r \frac{dx_r}{ds_r} ds_r,$$

$$\frac{dy}{ds} ds\Sigma_1^n a_r = \Sigma_1^n a_r \frac{dy_r}{ds_r} ds_r.$$

But
$$\frac{dx_r}{ds_r}, \quad \frac{dy_r}{ds_r}$$

are the same for all values of r, on account of the tangents being parallel. Hence we get

$$\left.\begin{aligned} \frac{dx}{ds} ds\, \Sigma_1^n a_r &= \frac{dx_r}{ds_r} \Sigma a_r ds_r, \\ \frac{dy}{ds} ds\, \Sigma_1^n a_r &= \frac{dy_r}{ds_r} \Sigma a_r ds_r, \end{aligned}\right\} \tag{57}$$

from which, by squaring, adding, and then extracting the square root, we obtain

$$ds\, \Sigma_1^n a_r = \Sigma a_r ds_r; \tag{58}$$

and this, by integration, evidently gives the result stated above.

It may be observed that it also follows from these equations that the tangent to the locus has the same direction as the tangents at the points on the generating curves.

It is shown in treatises on plane curves that if we draw all the tangents parallel to a given direction, then the centre of gravity of the points of contact is a fixed point. Hence, by the theorem of this Article, if all the tangents of a curve be drawn parallel to a given direction, the sum of the arcual distances of the points of contact from a given point of the curve is constant.

212. We now give Steiner's theorem on the rectification of roulettes which we referred to in Art. 164. If a curve roll on a right line, the length of the arc of the roulette, described by a point P invariably connected with the rolling curve is equal to that of the corresponding arc of the pedal taken with regard to P.

We have seen already in the Article referred to that if x, y are the co-ordinates of P, then

$$dx = p\,d\omega, \quad dy = dp\,;$$

hence, we have

$$dx^2 + dy^2 = dp^2 + p^2 d\omega^2,$$

which is evidently equivalent to the statement just made, as p, ω are the polar co-ordinates of a point on the pedal. From this result we may notice that it follows that the length of the roulette described by a focus of an ellipse rolling on a right line is equal to an arc of the auxiliary circle.

Again, we see that the length of the arc of the cycloid can be obtained from that of the cardioid; for the latter curve is the pedal of a circle with regard to a point on the circumference.

213. In the motion of a rigid lamina in a plane, if we are given the lengths of the envelopes of two lines of the figure, we can find the length of the envelope of any other line.

Let p_1, p_2, p_3 be the perpendiculars from an arbitrary origin on three lines of the figure, then we have

$$ap_1 + bp_2 + cp_3 = 2\Delta, \tag{59}$$

where a, b, c are the sides, and Δ the area of the triangle formed by the lines. Hence, differentiating twice with regard to ω, the angle through which the figure turns, we get

$$\frac{ad^2p_1}{d\omega^2} + \frac{bd^2p_2}{d\omega^2} + \frac{cd^2p_3}{d\omega^2} = 0;$$

whence, from (59), by addition, and putting

$$\frac{ds_1}{d\omega} \text{ for } p_1 + \frac{d^2p_1}{d\omega^2}, \text{ \&c.,}$$

we obtain

$$\frac{ads_1}{d\omega} + \frac{bds_2}{d\omega} + \frac{cds_3}{d\omega} = 2\Delta.$$

Integrating, then, we have

$$as_1 + bs_2 + cs_3 = 2\Delta\omega + C.$$

If the figure return to its original position after making a complete revolution, we take ω between the limits 2π and 0, and there results

$$as_1 + bs_2 + cs_3 = 4\pi\Delta, \tag{60}$$

where s_1, s_2, s_3 are now the perimeters of the curves enveloped by the lines.

From (60) it is easy to see that lines of the figure which envelop curves of equal perimeter are all tangents to a circle. For, taking three such lines, we have

$$(a + b + c) L = 4\pi\Delta,$$

where L is the common length. Now if r is the radius of the inscribed circle,

$$(a + b + c) r = 2\Delta;$$

therefore $L = 2\pi r$, from which the result just stated follows at once. This theorem is due to Mr. W. S. M'Cay.

Examples.

1. Show that the integral

$$\int \frac{dx}{x^2} \sqrt{(x^4 + 2x^2 \cos \omega + 1)}$$

can be represented by an arc of an hyperbola.

2. If s is the length of the cycloid represented by the equations

$$x = a(1 - \cos\theta), \quad y = a(\theta + \sin\theta),$$

show that $s^2 = 8ax$.

3. If s is the arc of the curve $(m + 1) y = x^{m+1}$, show that

$$s = \frac{x\sqrt{(1 + x^{2m})}}{m+1} + \frac{m}{m+1}\int \frac{dx}{\sqrt{(1 + x^{2m})}}.$$

4. Show that all the curves comprised in the equation $cy^m = x^{m\pm1}$, where m is an integer, are rectifiable by means of the elementary integrals. Show that this is also the case if the axes are oblique.

5. If s is the length of an arc of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{m}} + \left(\frac{y}{b}\right)^{\frac{2}{m}} = 1,$$

show that

$$s = \frac{m}{2}\int \sqrt{\{a^2 t^{m-2} + b^2(1 - t)^{m-2}\}}\, dt,$$

where $$x = at^{\frac{m}{2}}, \quad y = b(1 - t)^{\frac{m}{2}},$$

6. If s is the arc of the curve whose equation in oblique co-ordinates is

$$\left(\frac{x}{a}\right)^{\frac{1}{m}} + \left(\frac{y}{b}\right)^{\frac{1}{m}} = 1,$$

show that

$$s = m\int \sqrt{\{a^2 t^{2m-2} + b^2(1-t)^{2m-2} - 2ab\cos\omega\,(t-t^2)^{m-1}\}}\,dt,$$

where $$x = at^m, \quad y = b(1-t)^m.$$

7. Given a curve defined by the equations

$$x = \tfrac{2}{5}t^5 - 2t^3, \quad y = t^2 - \tfrac{3}{2}t^4,$$

show that its length measured from the origin is

$$\tfrac{2}{5}\{(1+t^2)^{\frac{5}{2}} - 1\}.$$

8. If a curve be given by the equations

$$x \pm iy = \int (t \pm ia)^m (t \mp ia)^n\,dt,$$

where m and n are both even, or both odd positive integers, show that the arc is an algebraic function of t.

9. If s is the arc of the curve given by the equations

$$x = 2at + \tfrac{2}{3}t^3, \quad y = (a^2-1)t + \tfrac{2}{3}at^3 + \tfrac{1}{5}t^5,$$

show that $$s = y + 2t.$$

10. If a curve be such that the length of the tangent measured from the point of contact to a fixed line is constant, show that the length of an arc is proportional to the logarithm of the ratio of the perpendiculars from its extremities on a fixed line.

11. If s is the arc of the curve, $y = ce^{\frac{x}{a}}$, show that

$$s = \sqrt{(a^2+y^2)} + a\log\left\{\frac{\sqrt{(a^2+y^2)} - a}{y}\right\}.$$

12. Show that the length of the curve $y = \log(1-x^2)$ measured from the origin is

$$\log\left(\frac{1+x}{1-x}\right) - x.$$

13. Show that the length of a loop of the curve $r^m = a^m \cos m\theta$

$$\frac{a\sqrt{\pi}}{m} \frac{\Gamma\left(\frac{1}{2m}\right)}{\Gamma\left(\frac{m+1}{2m}\right)}.$$

14. Show that the lengths of the loops of the positive and negative pedals of $r^m = a^m \cos m\theta$ are in a constant numerical ratio.

15. Show that the length of the curve $r = a\theta$ is

$$\frac{r\sqrt{(a^2 + r^2)}}{2a} + \frac{a}{2} \log \left\{\frac{r + \sqrt{(a^2 + r^2)}}{a}\right\}.$$

16. If s is the arc of a curve which is given as the envelope of the line

$$\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0,$$

show that
$$s = t - \int \frac{\alpha d\alpha}{\sqrt{(\alpha^2 + \beta^2)}}$$

where t is the length of the tangent measured from the point of contact to the axis of x. Also show that the similar formula for the axis of y is

$$s = -t' + \int \frac{\beta d\beta}{\sqrt{(\alpha^2 + \beta^2)}}.$$

17. Show that the arc of the envelope of the line

$$x \cos \omega + y \sin \omega - \phi'(\omega) = 0 \text{ is } \phi(\omega) + \phi''(\omega).$$

18. If σ is the arc of the curve enveloped by lines drawn, making a constant angle α with a given curve, show that

$$\sigma = \rho \sin \alpha \pm s \cos \alpha + C,$$

where ρ, s are the radius of curvature and the length of the arc, respectively, at the point on the given curve.

19. Show that the arc of the ellipse or hyperbola can be expressed as follows:

$$s = \int \frac{a^2 b^2 d\omega}{(a^2 \cos^2 \omega \pm b^2 \sin^2 \omega)^{\frac{3}{2}}}.$$

20. If a curve in p, ω co-ordinates be generated from another in polar co-ordinates by the transformation

$$r^2 = 2ap, \quad 2\theta = \omega,$$

find the relation between the arc of the first curve and the area of the second.

21. Show, by the transformation given in the preceding example, that the problem of finding the area of the Cassinian oval is the same as that of the rectification of the ellipse or hyperbola.

22. If Σ is the area of a closed curve described by the point x', y', and P the perimeter of the closed curve enveloped by the line

$$ax(x'^2 - y'^2) + 2ax'y'y - (x'^2 + y'^2)^2 = 0,$$

show that $$4\Sigma = aS.$$

23. Show that the arc of an ellipse can be expressed by the integral

$$\int \frac{r^2 dr}{\sqrt{\{(a^2 - r^2)(r^2 - b^2)\}}}$$

where r is the semidiameter parallel to the tangent.

24. If an endless string be passed round the circumference of an ellipse, and be kept stretched by a moving pencil, show that the pencil will trace out a confocal ellipse.

25. If from any point P of the ellipse

$$\frac{x^2}{a^2 + h^2} + \frac{y^2}{b^2 + h^2} - 1 = 0,$$

tangents PT, PT' be drawn to the confocal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that

$$PT + PT' - \text{arc } TT' = 2h\sqrt{\left(\frac{a^2 + h^2}{b^2 + h^2}\right)} - 2a\,E(\phi),$$

where $\tan\phi = h/b$.

26. If x, y are connected with the variables μ, ν, by the equations

$$cx = \mu\nu, \quad cy = \sqrt{\{(\mu^2 - c^2)(c^2 - \nu^2)\}},$$

show that the differential equation

$$\frac{d\mu}{\sqrt{(a^2-\mu^2)}} \pm \frac{d\nu}{\sqrt{(a^2-\nu^2)}} = 0$$

represents a series of circles having double contact with the conic $\mu = a$.

27. In the same case, show that the differential equation

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-c^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2-\nu^2)(c^2-\nu^2)\}}} = 0$$

represents another system of circles having double contact with the same conic.

28. If the portion of the tangent of an ellipse intercepted between two fixed tangents is a minimum, show that its extremities are equidistant from the centre of the curve.

29. If a curve be determined by the equations

$$x = \frac{az}{1+z^4}, \quad y = \frac{az^3}{1+z^4},$$

show that the element of the arc is

$$\frac{a\,dz}{\sqrt{(1+z^4)}}.$$

30. If m is any integer, show that the inverse of the curve $cy^m = x^{m\pm1}$ with regard to any point can be rectified by means of the elementary integrals.

31. Show that the arc of the inverse of the curve $cy^2 = x^3$ with regard to the point $-4c/27,\ 0$ is proportional to

$$\int \frac{d\theta}{(9\theta+4)^{\frac{3}{2}}(9\theta+1)},$$

where $x = c\theta,\ y = c\theta^{\frac{3}{2}}$ for the given curve.

32. If s is the arc of the envelope of the circle

$$\cos 3\omega(x^2+y^2) - b(x\cos\omega + y\sin\omega) = 0,$$

show that $$s = \frac{2b\sqrt{2}}{3}\tan^{-1}(2\sqrt{2}\sin 3\omega).$$

33. Show that the arc of the envelope of the circle

$$\cos\tfrac{1}{2}\omega\,(x^2+y^2)-b\,(x\cos\omega+y\sin\omega)=0 \text{ is}$$

$$\frac{b\sqrt{3}}{2}\log\left\{\frac{2+\sqrt{3}\sin\frac{1}{2}\omega}{2-\sqrt{3}\sin\frac{1}{2}\omega}\right\}.$$

34. Show that the sectorial area of Serret's curves varies as the difference of the areas of the generating triangles at the corresponding points.

35. If a curve be obtained by eliminating ϕ between the equations

$$r=a\sqrt{(1-m^2)}+\frac{a}{\sin\phi},$$

$$\cos m\theta=\frac{1+\sqrt{(1-m^2)}\sin\phi}{\sqrt{(1-m^2)}+\sin\phi},$$

show that ϕ is the angle which the radius vector makes with the curve, and that $s=b\cot\phi$, where s is the length of the arc.

36. If a curve be obtained by eliminating ϕ between the equations

$$r^{2n}=\sin^2\phi+m^2\cos^2\phi,\quad m\tan\left(\frac{\phi-n\theta}{m}\right)=\tan\phi,$$

show that ϕ is the angle which the radius vector makes with the tangent, and that

$$s=\int\frac{\sqrt{(m^2-1)}\,dr}{\sqrt{(r^{2n}-1)}},\quad\text{or}\quad\int\frac{\sqrt{(1-m^2)}\,dr}{\sqrt{(1-r^{2n})}},$$

where s is the length of the arc.

37. If s is the arc of the curve,

$$r^{2n}-2a^nr^n\cos n\theta+a^{2n}-b^{2n}=0,$$

show that $$ds=\frac{2b^nr^n\,dr}{\sqrt{\{-r^{4n}+2(a^{2n}+b^{2n})r^{2n}-(a^{2n}-b^{2n})^2\}}}.$$

38. Show that the element of the arc of the curve whose polar equation is given in the preceding example can be written in the form

$$\frac{b^n\,d\phi}{(a^{2n}-2a^nb^n\sin n\phi+b^{2n})^{\frac{n-1}{2n}}},$$

by assuming $$r^{2n}=a^{2n}-2a^nb^n\sin n\phi+b^{2n}.$$

2 X

39. Show that the polar equations of Serret's curves, corresponding to the cases of $n = 2$, and $n = 1/2$ in Art. 207, are

$$4\rho^6 + 27\rho^4 - 12\rho^2 + 1 - 12\sqrt{3}\rho^5 \cos\theta = 0,$$

$$\rho^6 + 6\rho^2 - 2 - 3\sqrt{3}\rho^4 \cos 2\theta = 0,$$

respectively, and hence that the arcs of these curves of the sixth degree are expressible by elliptic integrals of the first kind whose moduli are $\sqrt{2}/\sqrt{3}, 1/\sqrt{3}$.

40. If on the tangent at a point P of a given curve we measure out a constant length PQ equal to a, show that the arc of the locus of Q is $\int \sqrt{(\rho^2 + a^2)}\, d\omega$, where ρ is the radius of curvature of the given curve.

41. If the given curve in the preceding example is a hypocycloid, show that the arc of the locus is equal to that of an ellipse.

42. Show that the arc and area of the curve $r^3 \cos 3\theta = a^3$ are expressible by means of the same integrals as the area and arc of the curve $r^3 = a^3 \cos 3\theta$, respectively.

43. A circle passing through the origin and a fixed point on the curve $r^3 = a^3 \cos 3\theta$ meets the curve again in A, B; show that the middle point of the arc AB is fixed.

44. If from the equation of a curve in rectangular co-ordinates we form another in polar co-ordinates by the transformation $r = y$, $rd\theta = dx$, show that the lengths of corresponding arcs of the curves are equal, and that the area $\int y\,dx$ of the former curve is equal to double the sectorial area of the latter.

45. If a closed curve be such that the distances r_1, r_2, &c., of any point thereon from certain fixed points are connected by the relation $l_1 r_1 + l_2 r_2 + \text{\&c.} = l$, show that the perimeters s_1, s_2, &c., of the pedals with regard to these points are connected by the relation

$$l_1 s_1 + l_2 s_2 + \text{\&c.} = 2\pi l.$$

46. Let

$$\varpi_1 = x \cos\omega_1 + y \sin\omega_1 - p_1 = 0,$$

$$\varpi_2 = x \cos\omega_2 + y \sin\omega_2 - p_2 = 0, \text{ \&c.},$$

be the tangents to certain curves, all inclined to each other at constant angles. Show that the arc of the envelope of the line $l_1\varpi_1 + l_2\varpi_2 + \text{\&c.} = 0$ can be expressed linearly in terms of arcs of each of the curves.

47. If a curve be determined by the equations

$$x = \alpha^3 - 3\alpha\beta^2 - 3\alpha, \quad y = \beta^3 - 3\beta\alpha^2 + 3\beta,$$

where
$$(\alpha^2 + \beta^2)^2 - 2(\alpha^2 - \beta^2) = 0,$$

show that its arc is equal to that of a lemniscate.

48. If a point x', y' be connected with a point x, y by the equations

$$x' = \frac{x(x^2 + y^2 + c^2)}{x^2 + y^2}, \quad y' = \frac{y(x^2 + y^2 - c^2)}{x^2 + y^2},$$

show that
$$ds' = Pds,$$

where
$$P = \frac{\sqrt{\{(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4\}}}{(x^2 + y^2)}.$$

49. If a curve in polar co-ordinates be transformed by the substitution $r' = r^n$, $\theta' = n\theta$, show that $ds' = nr^{n-1}ds$.

50. If a curve be transformed into another by the substitution $\mu + \sqrt{(\mu^2 - c^2)} = r$, $\nu = c\cos\theta$, where μ, ν are elliptic co-ordinates, and r, θ polar co-ordinates, show that

$$\frac{ds_{\mu,\nu}}{\sqrt{(\mu^2 - \nu^2)}} = \frac{ds_{r,\theta}}{r}.$$

51. Show that the arc of the curve, whose equation in elliptic co-ordinates is $\mu + \sqrt{(\mu^2 - c^2)} = m\sqrt{(\mu^2 - \nu^2)}$, is equal to an arc of a Cassinian oval.

52. Show that the arc of the curve, whose equation in elliptic co-ordinates is

$$l\mu + m\sqrt{(\mu^2 - c^2)} = \sqrt{(\mu^2 - \nu^2)},$$

can be expressed in terms of two arcs of a bicircular quartic with a centre.

MISCELLANEOUS EXAMPLES.

1. $$\int \frac{(2x+1)\,dx}{(x^2+2x+5)(x^2+x+1)(x^2+1)} = \frac{1}{130}\log(x^2+2x+5)$$

$$+\frac{33}{260}\tan^{-1}\left(\frac{x+1}{2}\right) + \frac{5}{26}\log(1+x+x^2) - \frac{3\sqrt{3}}{13}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{1}{5}\log(1+x^2) + \frac{3}{10}\tan^{-1}x.$$

2. $$\int \frac{x^3\,dx}{(a+bx)^4} = \frac{11a^3 + 27a^2bx + 18ab^2x^2}{6b^4(a+bx)^3} + \frac{1}{b^4}\log(a+bx).$$

3. $$\int \frac{x^3\,dx}{(a+bx)^5} = -\frac{(a^3 + 4a^2bx + 6ab^2x^2 + 4b^3x^3)}{4b^4(a+bx)^4}.$$

4. $$\int \frac{x^2\,dx}{(1+x^2)^4} = \left(\frac{x^5}{16} + \frac{x^3}{6} - \frac{x}{16}\right)\frac{1}{(1+x^2)^3} + \frac{1}{16}\tan^{-1}x.$$

5. $$\int \frac{x^3\,dx}{(a+bx+cx^2)^2} = \frac{a^2 + 2bx}{2c^2(a+bx+cx^2)} + \frac{1}{2c^3}\log(a+bx+cx^2)$$

$$-\frac{ab(2cx+b)}{2c^3(4ac-b^2)(a+bx+cx^2)} - \frac{b(6ac-b^2)}{2c^2(ac-b^2)}\int\frac{dx}{a+bx+cx^2}.$$

6. $$\int (a+bx^2)^{\frac{5}{2}}\frac{dx}{x^{10}} = \frac{(2bx^2-7a)}{63a^2x^9}(a+bx^2)^{\frac{7}{2}}.$$

7. $$3(p-1)(ab^2-4a^2c)\int\frac{dx}{X^p}$$

$$= \frac{bcx^4 + (b^2-2ac)x}{X^{p-1}} + 2bc(3p-5)\int\frac{x^3\,dx}{X^{p-1}}$$

$$+\{3(p-1)(b^2-4ac) + 2ac - b^2\}\int\frac{dx}{X^{p-1}},$$

where $$X = a + bx^3 + cx^6.$$

8. $$\int (\sin\theta)^7\,d\theta = \frac{1}{448}\cos 7\theta - \frac{7}{320}\cos 5\theta + \frac{7}{64}\cos 3\theta - \frac{35}{64}\cos\theta.$$

9. $$\int \sin^3\theta\cos^5\theta\,d\theta = \frac{1}{128}\left\{\frac{1}{8}\cos 8\theta + \frac{1}{3}\cos 6\theta - \frac{1}{2}\cos 4\theta - 3\cos 2\theta\right\}.$$

10. $$\int \sin^4\theta\cos^5\theta\,d\theta = \frac{1}{256}\left\{\frac{1}{9}\sin 9\theta + \frac{1}{7}\sin 7\theta - \frac{4}{5}\sin 5\theta - \frac{4}{3}\sin 3\theta + 6\sin\theta\right\}.$$

11. $$\int \frac{\sin^6\theta\,d\theta}{\cos^2\theta} = \frac{1}{8\cos\theta}(15\sin\theta - 5\sin^3\theta - 2\sin^5\theta) - \frac{15}{8}\theta.$$

12. $$\int \frac{d\theta}{\sin^2\theta\cos^7\theta} = \frac{1}{48\sin\theta\cos^6\theta}\{8 + 14\cos^2\theta + 35\cos^4\theta - 105\cos^6\theta\} + \frac{35}{16}\log(\sec\theta + \tan\theta).$$

13. $$\int \theta^5\cos\theta\,d\theta = (\theta^5 - 20\theta^3 + 120\,\theta)\sin\theta + 5\,(\theta^4 - 12\theta^2 + 24)\cos\theta.$$

14. $$\int \frac{x\sin^{-1}x\,dx}{(1-x^2)^{\frac{3}{2}}} = \frac{\sin^{-1}x}{\sqrt{(1-x^2)}} + \frac{1}{2}\log\left(\frac{1-x}{1+x}\right).$$

15. To prove Weierstrass's transformation in elliptic integrals, namely,

$$\frac{x\,dy - y\,dx}{\sqrt{U}} = \frac{1}{2}\frac{dz}{\sqrt{(-4z^3 + Sz - T)}},$$

where $$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

$$S = ae - 4bd + 3c^2,$$

$$T = ace + 2bcd - ad^2 - eb^2 - c^3,$$

and $z = H/U$, H being equal to

$$\frac{1}{144}\left\{\frac{d^2U}{dx^2}\frac{d^2U}{dy^2} - \left(\frac{d^2U}{dx\,dy}\right)^2\right\}$$

$$= (ac - b^2)x^4 - 2(ad - bc)x^3y + (3c^2 - ae - 2bd)x^2y^2 - 2(be - cd)xy^3 + (d^2 - ce)y^4.$$

We have

$$dz = \frac{UdH - HdU}{U^2}$$

$$= \frac{1}{4U^2}\left(\frac{xdU}{dx} + \frac{ydU}{dy}\right)\left(\frac{dH}{dx}dx + \frac{dH}{dy}dy\right)$$

$$- \frac{1}{4U^2}\left(\frac{xdH}{dx} + \frac{ydH}{dy}\right)\left(\frac{dU}{dx}dx + \frac{dU}{dy}dy\right)$$

$$= \frac{1}{4U^2}\left(\frac{dU}{dx}\frac{dH}{dy} - \frac{dU}{dy}\frac{dH}{dx}\right)(xdy - ydx);$$

so that if we put $$\frac{dU}{dx}\frac{dH}{dy} - \frac{dU}{dy}\frac{dH}{dx} = 8J,$$

we get $$dz = \frac{2J}{U^2}(xdy - ydx).$$

Now it is shown in treatises on algebra that

$$J^2 = -4H^3 + SHU^2 - TU^3.$$

Hence we have

$$\frac{xdy - ydx}{\sqrt{U}} = \frac{U^{\frac{3}{2}}dz}{2J} = \frac{U^{\frac{3}{2}}dz}{2\sqrt{(-4H^3 + SHU^2 - TU^3)}},$$

which gives the required result by putting $H = Uz$.

16. Reduce the integral

$$\int \frac{lU + mH}{l'U + m'H}\,\frac{(xdy - ydx)}{\sqrt{U}},$$

by Weierstrass's transformation, where U, H have the same meaning as in the preceding example.

17. U and V are binary quartics, such that V is of the form

$$aP^2 + bQ^2 + 2hPQ,$$

where P, Q are quadratic factors of U; show that

$$\int \frac{\sqrt{U}}{V}(xdy - ydx)$$

can be made to depend upon two elliptic integrals of the third kind with the same modulus and different parameters.

18. U is a binary quartic and V a quadratic; show that

$$\int \frac{\left(\frac{dV}{dx}\frac{dU}{dy} - \frac{dV}{dy}\frac{dU}{dx}\right)}{U + V^2} \frac{(xdy - ydx)}{\sqrt{U}} = 4\tan^{-1}\left(\frac{\sqrt{U}}{V}\right),$$

and hence deduce a relation connecting four elliptic integrals of the third kind.

19. P, Q are quadratic functions of x, y; show that

$$\int \frac{4PQ + xy\left(\frac{dP}{dx}\frac{dQ}{dy} - \frac{dP}{dy}\frac{dQ}{dx}\right)}{Py^2 + Qx^2} \frac{(xdy - ydx)}{\sqrt{(PQ)}} = 4\tan^{-1}\left\{\frac{y}{x}\sqrt{\left(\frac{P}{Q}\right)}\right\},$$

and hence deduce a relation connecting four elliptic integrals of the third kind.

20. To show that

$$\int \frac{xdy - ydx}{U^{\frac{2}{3}}},$$

where U is a binary cubic, can be expressed by an elliptic integral of the first kind with a numerical modulus.

If we apply the linear transformation as in Art. 20, the integral is found to retain the same form. If we suppose then the coefficients of y^3 and x^2y to vanish in the new expression for U, the integral may be written

$$\int \frac{xdy - ydx}{\{x(\alpha x^2 \pm \gamma y^2)\}^{\frac{2}{3}}},$$

which, when we take $x = 1$, and put $\alpha \pm \gamma y^2 = z^3$, becomes

$$\frac{3}{2}\int \frac{dz}{\sqrt{\{\pm\gamma(\alpha - z^3)\}}},$$

and this comes under either Ex. 1 or Ex. 2, p. 133.

Or thus: Putting with the notation of Ex. 40, p. 38, $z = \frac{H}{U^{\frac{2}{3}}}$, we have

$$3UH\frac{dz}{z} = 3UdH - 2HdU = 3G(xdy - ydx);$$

since $G^2 + 4H^3 = \Delta U^2$, where Δ is the discriminant of U, namely,

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.$$

We find thus

$$\int \frac{xdy - ydx}{U^{\frac{2}{3}}} = \int \frac{dz}{\sqrt{(\Delta - 4z^3)}}.$$

21. If the relation

$$(x-a)(x-b)(x-c) + \lambda(x-x_1)(x-x_2)(x-x_3) = (mx+n)^3$$

is identically satisfied for all values of x, show, by the method of Art. 99, that

$$\Sigma_1^3 \frac{dx_r}{\{(x_r-a)(x_r-b)(x_r-c)\}^{\frac{2}{3}}} = 0.$$

22. Show that the relation

$$(b-c)\sqrt[3]{\{(a-x)(a-y)(a-k)\}} + (c-a)\sqrt[3]{\{(b-x)(b-y)(b-k)\}}$$
$$+ (a-b)\sqrt[3]{\{(c-x)(c-y)(c-k)\}} = 0,$$

where k is an arbitrary constant, is an integral of

$$\frac{dx}{\{(x-a)(x-b)(x-c\}^{\frac{2}{3}}} + \frac{dy}{\{(y-a)(y-b)(y-c)\}^{\frac{2}{3}}} = 0.$$

23. With the notation of Ex. 15, show that

$$\int \frac{(xdy - ydx)}{\sqrt[3]{J}} = \frac{1}{2}\int \frac{dz}{(4z^3 - Sz + T)^{\frac{2}{3}}}.$$

24. Show that the integral of the differential equation

$$\frac{dx}{\sqrt{\{(x-a)(x-b)(x-c)(x-d)\}}} \pm \frac{dy}{\sqrt{\{(y-a)(y-b)(y-c)(y-d)\}}} = 0$$

can be written in any one of the four forms

$$l\sqrt{\{(a-x)(a-y)\}} + m\sqrt{\{(b-x)(b-y)\}} + n\sqrt{\{(c-x)(c-y)\}} = 0,$$

where

$$l^2(a-b)(a-c)(a-d) + m^2(b-a)(b-c)(b-d) + n^2(c-a)(c-b)(c-d) = 0.$$

25. If $\text{sn}^2 u_1$, $\text{sn}^2 u_2$, $\text{sn}^2 u_3$, $\text{sn}^2 u_4$ are the roots of the equation

$$x(\alpha + \beta x)^2 - (\gamma + \delta x)^2 (1-x)(1-k^2 x) = 0,$$

show, by the method of Art. 99, that

$$u_1 \pm u_2 \pm u_3 \pm u_4 = 2mK + (2n+1)\, iK'.$$

26. If $\operatorname{sn} u_1$, $\operatorname{sn} u_2$, $\operatorname{sn} u_3$, $\operatorname{sn} u_4$ are the roots of the equation

$$(\alpha + \beta x)^2 (1 - x^2) - (\gamma + \delta x)^2 (1 - k^2 x^2) = 0,$$

show that $$u_1 + u_2 + u_3 + u_4 = (4m + 2) K + 2niK'.$$

27. To show that

$$\int \frac{(\alpha x + \beta y)(x dy - y dx)}{\sqrt{U}},$$

where U is a binary sextic, can be expressed by elliptic integrals of the first kind, if U has its roots forming a system in involution.

In the case in question U is the product of three quadratic factors which are connected by a linear relation, and, therefore, if we transform two of the quadratics, as in Art. 20, so as to involve the squares only of the new variables, the third also will involve these squares. The integral being transformed thus takes the form

$$\int \frac{(\alpha' x + \beta' y)(x dy - y dx)}{\sqrt{\{(lx^2 + my^2)(l' x + m' y^2)(l'' x^2 + m'' y^2)\}}},$$

which, if we put $y = x\sqrt{t}$, becomes

$$\tfrac{1}{2}\alpha' \int \frac{dt}{\sqrt{\{t(l + mt)(l' + m't)(l'' + m''t)\}}} + \tfrac{1}{2}\beta' \int \frac{dt}{\sqrt{\{(l + mt)(l' + m't)(l'' + m''t)\}}};$$

but these are evidently two elliptic integrals of the first kind.

It is to be observed that the reduction to elliptic integrals with real moduli obtains when the new variables are capable of being expressed in a real manner in terms of the old. If this is not the case the reduction is useless, at least for purposes of calculation, as the moduli are then imaginary.

28. Express $$\int \frac{d\theta}{(1 - c^2 \sin^2 \theta)^{\frac{1}{2}}}$$

by means of elliptic integrals of the first kind.

29. Given $$\tan\left(\frac{\psi - \phi}{2}\right) = \left(\frac{1 - m}{m}\right) \tan \phi,$$

show that $$F_k(\phi) = \frac{m}{2 - m} F_h(\psi),$$

where $$k^2 = \frac{2m - 1}{m^3 (2 - m)} \quad h = k^3 m^4$$

30. Show that

$$(a+b-c-d)xy+(cd-ab)(x+y)+ab(c+d)-cd(a+b)=0,$$

and two other similar relations are integrals of the equation

$$\frac{dx}{\sqrt{\{(x-a)(x-b)(x-c)(x-d)\}}}+\frac{dy}{\sqrt{\{(y-a)(y-b)(y-c)(y-d)\}}}=0.$$

31. $\displaystyle\int_{-\infty}^{\infty} f(\sin x, \cos x) \log\left(1+\frac{a^2}{x^2}\right) dx$

$$=\int_0^{2\pi} f(\sin x, \cos x) \log\left\{1+\frac{(e^a-1)^2}{4e^a \sin^2 \frac{1}{2}x}\right\} dx.$$

This relation may be obtained by a process exactly similar to that used in Ex. 6, p. 170.

32. $$\int_0^{\infty} u \log \cos \theta\, dx = \int_0^1 u \log\left(\frac{\sin 2\theta}{2\theta}\right) dx,$$

where $$u=f(\sin 2\pi x, \cos 2\pi x),$$

$$\theta=\frac{a}{2^x}.$$

33. $$\int_0^{\infty} u \log(1+e^{-cz})\, dx = \int_0^1 u \log \frac{1}{(1-e^{-cz})}\, dx,$$

where $$u=f(\sin 2\pi x, \cos 2\pi x),$$

$$z=2^x.$$

34. $$\int_0^1 \frac{x^{a-1}(1+x-2x^a)\,dx}{1-x^2}=\log 2.$$

35. $$\int_0^1 \frac{(\log x)^2\,dx}{\sqrt{(1-x^2)}}=\frac{\pi^3}{24}+\frac{\pi}{2}(\log 2)^2.$$

36. $$\int_0^1 \frac{(x^{a-1}-x^{n-a-1})(\log x)^2\,dx}{1-x^n}=\frac{2\pi^3 \cos\left(\frac{a\pi}{n}\right)}{n^3 \sin^3\left(\frac{a\pi}{n}\right)}.$$

37. If
$$I = \int_0^\infty e^{-x} \log \frac{1}{x}\, dx,$$

$$I' = \int_0^\infty e^{-x} (\log x)^2\, dx,$$

show that
$$I' = I^2 + \frac{\pi^2}{6}.$$

38. Show that

$$\int_0^{\frac{\pi}{4}} \log \sin\theta\, d\theta = -\frac{1}{2}\left(C_2 + \frac{\pi}{2}\log 2\right),$$

$$\int_0^{\frac{\pi}{4}} \log \sin\theta\, d\theta = \frac{1}{2}\left(C_2 - \frac{\pi}{2}\log 2\right),$$

where
$$C_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c.$$

39. Show that
$$\int_0^{\frac{\pi}{4}} \theta \tan\theta\, d\theta = \frac{1}{2}\left(C_2 - \frac{\pi}{4}\log 2\right),$$

$$\int_0^{\frac{\pi}{4}} \frac{\theta^2\, d\theta}{\cos^2\theta} = \frac{\pi^2}{16} + \frac{\pi}{4}\log 2 - C_2,$$

where C_2 has the same meaning as in the preceding example.

40.
$$\int_0^{\frac{1}{2}\pi} \frac{(\log \tan\theta)^2\, d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{\pi}{16ab}\left\{\pi^2 + 4\left(\log\frac{b}{a}\right)^2\right\}.$$

41. Show that

$$\int_0^\infty \frac{x^{n-1}\, dx}{(a + bx + cx^2)^n} = \frac{\sqrt{\pi}}{c^{\frac{1}{2}}(b + 2\sqrt{ac})^{n-\frac{1}{2}}} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)},$$

$$\int_0^\infty \frac{x^{n-\frac{3}{2}}\, dx}{(a + bx + cx^2)^n} = \frac{\sqrt{\pi}}{a^{\frac{1}{2}}(b + 2\sqrt{ac})^{n-\frac{1}{2}}} \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n)},$$

where a, b, c are positive. (Professor Cayley.)

42. $$\int_{-\infty}^{\infty} \frac{\cos m\left(x - \frac{a}{x}\right) dx}{1 + \left(x - \frac{a}{x}\right)^2} = \pi e^{-m}.$$

43. Show that

$$\int_0^{\frac{1}{2}\pi} \cos(x \sin\theta)\, d\theta = \frac{\pi}{2}\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \&c.\right\}$$

44. Given $$y = \int_0^{\pi} e^{ax\cos\theta}\, d\theta,$$

show that $$\frac{xd^2y}{dx^2} + \frac{dy}{dx} - a^2xy = 0.$$

45. If

$$(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}} = 1 + \alpha P_1 + \alpha^2 P_2 + \ldots + \alpha^r P_r + \&c.,$$

show that $$\int_{-1}^{1} P_m P_n dx = 0,$$

except when $$m = n,$$

and then $$\int_{-1}^{1} P_m{}^2 dx = \frac{2}{2m+1}.$$

46. If the point x, y lies outside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that $$\int_0^{2\pi} \frac{d\theta}{\sqrt{\{(x - a\cos\theta)^2 + (y - b\sin\theta)^2\}}} = \frac{4K}{\sqrt{(\rho\rho')}},$$

where the modulus is equal to the sine of half the angle between the tangents drawn from x, y, and ρ, ρ' are the distances of the same point from the foci.

47. If $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 > 0,$$

show that

$$\int_0^{2\pi} \frac{d\theta}{(x - a\cos\theta)^2 + (y - b\sin\theta)^2} = \frac{2\pi\sqrt{\{(a^2 + \lambda_1)(b^2 + \lambda_1)\}}}{\lambda_1(\lambda_1 - \lambda_2)};$$

where λ_1, λ_2 are the greatest and least roots, respectively, of the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} - 1 = 0.$$

48. If
$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \equiv S > 0,$$
show that
$$\int_0^{2\pi} \frac{d\theta}{\{(x - a\cos\theta)^2 + (y - b\sin\theta)^2\}^2} = \frac{\pi(a^2 + b^2)}{a^3 b^3 S^2} + \frac{4\pi}{abS^3}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right).$$

49. If r is the distance of the element ds of a circle from a fixed point P, show that
$$\int \frac{ds}{r^6} = \frac{2\pi a(\delta^4 + 4a^2\delta^2 + a^4)}{(\delta^2 \sim a^2)^5},$$
where a is the radius, δ the distance of P from the centre, and the integral is taken throughout the entire perimeter.

50. If each element of a finite right line is divided by its distance from a fixed point, show that the sum of these quantities is equal to
$$\log\left(\frac{r + r' + \delta}{r + r' - \delta}\right),$$
where δ is the length of the line, and r, r' the distances of its extremities from the fixed point.

51. If S is the area cut off from the lemniscate $r^2 = 2c^2\cos 2\theta$ by a tangent of the curve at a point P, show that
$$S = \frac{(c^4 - r^4)^{\frac{3}{2}}}{c^4}.$$

52. A bicircular quartic consists of two ovals, one within the other; if the area which the tangent at a point P of the inner oval cuts off from the outer one is a maximum or a minimum, show that the normal at P passes through the centre of the generating conic.

53. A, B are two points on the curve
$$x^4 + y^4 + 2x^2y^2\cos 2\alpha = a^4,$$
such that the area AOB is given, where O is the origin. If concentric equilateral hyperbolae are described through A, B, show that they touch the curve
$$x^4 + y^4 + 2x^2y^2\cos 2\alpha - a^4 + \lambda(x^2 + y^2)^2 = 0.$$

54. If S is the area included between two normals of the parabola $y^2 - 4mx = 0$, the curve and its evolute, show that

$$S = 2m^2\{\alpha - \beta + \tfrac{2}{3}(\alpha^3 - \beta^3) + \tfrac{1}{5}(\alpha^5 - \beta^5)\},$$

where α, β are the tangents of the angles which the normals make with the axis of x.

55. A variable circle has double contact with a fixed curve at P, Q, and cuts off from another fixed curve a given area at the points A, B; show that the line PQ passes through the centre of gravity of the circular arc AB.

56. If Δ is the area between the ellipse

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and the tangents drawn to the curve from the point xy, show that

$$\Delta = ab\,(\sqrt{S} - \tan^{-1}\sqrt{S}).$$

57. If Δ is the similar value for the hyperbola

$$S \equiv 1 + \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0,$$

show that
$$\Delta = ab\sqrt{S} - \frac{1}{2}ab\log\left\{\frac{1+\sqrt{S}}{1-\sqrt{S}}\right\}.$$

58. A right line cuts off from two parabolae with parallel axes areas which are in a constant ratio; show that its envelope is a parabola.

59. Lemniscates of which the origin is the node are described to touch the equilateral hyperbola

$$x^2 - y^2 - a^2 = 0;$$

show that they cut off a constant area from

$$x^2 - y^2 - a'^2 = 0.$$

60. If the tangent at a point P of a curve meet an outer curve in A, B, so that the arc AB is given, show that AP/PB is inversely as the tangents to the outer curve at A, B. (MacCullagh.)

61. Two lines intersect on a curve and make angles with the normal whose sines are in a given ratio; find the relation connecting the arcs of their envelopes.

62. A Cartesian oval being written in the form

$$r^2 - 2r(c + a\cos\theta) + a^2 - b^2 = 0;$$

let s be the perimeter of the pedal of an oval with regard to the origin, and S the area of the pedal of the same oval with regard to the point

$$r = a, \quad \theta = 0,$$

then show that

$$2S = cs + \Sigma + \pi b^2.$$

where Σ is the area of the oval.

63. A circle of given radius turns about a fixed internal point; show that it is cut orthogonally in every position by curves represented by the equation

$$r^2 + k^2 - 2ar\cos\phi = 0,$$

where r is the length of the radius vector, and ϕ the angle it makes with the curve.

Show then also that the arc s of this curve is given by the equation

$$s = a\log(r^2 + k^2),$$

and that the curve is algebraic if k/a is a number.

64. A, B are two variable points on a given curve; show that if the arc AB is given, the locus of the middle point of the chord AB is cut orthogonally by the locus of the middle point of the chord of arcs whose point of bisection is fixed.

65. Three points, A, B, C, are taken on the axis of the hyperbola

$$3x^2 - y^2 = c^2,$$

so that

$$OA + OB + OC = 0, \quad OA^2 + OB^2 + OC^2 = 2c^2,$$

where O is the origin. If s_1, s_2, s_3 are the arcs of the inverses of an arc of the hyperbola with regard to A, B, C, respectively, show that

$$BC \cdot s_1 + CA \cdot s_2 + AB \cdot s_3 = 0.$$

66. If s_1, s_2 are the inverse arcs of an arc s of the curve

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 - 2k^2(x^2 + y^2 + c^2) = 0$$

with regard to the points $\pm c$, 0 and radius k, show that

$$s = s_1 + s_2.$$

67. Let A, B, C be the vertices of a triangle inscribed in the circle

$$x^2 + y^2 - (a+b)^2 = 0,$$

and circumscribed about the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

then show that the inverse arcs with respect to A, B, C of an arc of the curve

$$\{x^2 + y^2 + (a+b)^2\}^2 - 4(a^2x^2 + b^2y^2) = 0$$

are connected by a linear relation.

68. If ds is an element of the arc of a closed curve without inflexions, and r the distance of the same element from an internal point P, show that $\int \frac{ds}{r}$ taken throughout the entire perimeter has the same value for the curve itself and its polar reciprocal with regard to P.

69. If there be a system of curves given by the equation

$$\phi(\mu) + \psi(\mu) = \alpha,$$

where μ, ν are elliptic co-ordinates, as explained in Art. 189, and α is a parameter, show that they are all cut orthogonally by the system

$$\int \frac{d\mu}{\phi'(\mu)(\mu^2 - c^2)} - \int \frac{d\nu}{\psi'(\nu)(c^2 - \nu^2)} = \beta.$$

70. Show that the equations

$$\int \frac{d\mu}{\sqrt{(a^2 - \mu^2)}} \pm \int \frac{d\nu}{\sqrt{(a^2 - \nu^2)}} = \text{a constant},$$

$$\int \frac{\mu d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \int \frac{\nu d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = \text{a constant},$$

represent the two systems of circles having double contact with the conic $\mu = a$.

71. Show that the circles represented by the equation

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-c^2)\}}} \mp \frac{\nu d\nu}{\sqrt{\{(a^2-\nu^2)(c^2-\nu^2)\}}} = 0$$

are cut orthogonally by the curve

$$\int \frac{d\mu}{\mu}\sqrt{\left(\frac{\mu^2-a^2}{\mu^2-c^2}\right)} \mp \int \frac{d\nu}{\nu}\sqrt{\left(\frac{a^2-\nu^2}{c^2-\nu^2}\right)} = \text{a constant},$$

that is,

$$\{a\sqrt{(\mu^2-c^2)} + e\sqrt{(\mu^2-a^2)}\}\{a\sqrt{(c^2-\nu^2)} + c\sqrt{(a^2-\nu^2)}\}$$
$$= C\mu\nu\{\sqrt{(\mu^2-a^2)} + \sqrt{(\mu^2-c^2)}\}^e\{\sqrt{(a^2-\nu^2)} + \sqrt{(c^2-\nu^2)}\}^e,$$

where $$e = c/a.$$

Hence, also show, that if e is a number, the trajectory will be algebraic.

72. Show that the system of curves represented by the equation

$$\nu\{\mu + \sqrt{(\mu^2-c^2)}\} = a$$

are cut orthogonally by the system

$$\sqrt{(c^2-\nu^2)}\{\mu + \sqrt{(\mu^2-c^2)}\} = \beta.$$

Show also that these curves are circular cubics with nodes.

73. Show that the equations

$$\frac{d\mu}{\sqrt{\{(a^2-\mu^2)(\mu^2-a'^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2-\nu^2)(\nu^2-a'^2)\}}} = 0,$$

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-c^2)(a^2-\mu^2)(\mu^2-a'^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(c^2-\nu^2)(a^2-\nu^2)(a'^2-\nu^2)\}}} = 0,$$

$$\frac{d\mu}{\sqrt{\{(\mu^2-c^2)(\mu^2-a^2)(a'^2-\mu^2)\}}} \pm \frac{d\nu}{\sqrt{\{(c^2-\nu^2)(a^2-\nu^2)(a'^2-\nu^2)\}}} = 0$$

represent three systems of conics, respectively, having double contact with the confocal conics $\mu = a, \quad \nu = a'$.

74. Show that the system of Cartesian ovals

$$u + k\nu = mc,$$

where m is given and k variable, is cut orthogonally by the curve

$$\left\{\frac{\mu-c}{\mu+c}\right\}^n = C\left\{\frac{\mu^2-c^2}{c^2-\nu^2}\right\}.$$

75. Show that the arc s of the curve

$$\frac{x+iy}{\sqrt{\{c^2-(x+iy)^2\}}}+\frac{x-iy}{\sqrt{\{c^2-(x-iy)^2\}}}=2$$

is given by the equation

$$s=\int\frac{cdt}{(t^4+4)^{\frac{1}{2}}},$$

where $$t=\frac{c^2xy}{(x^2+y^2)^2-2c^2(x^2-y^2)+c^4}.$$

Hence show that the arc is expressible as that of the lemniscate of Bernouilli.

76. Show that the system of curves

$$\rho_1{}^l\rho_2{}^m\rho_3{}^n, \text{ \&c.} = k^{l+m+n+\text{\&c.}},$$

where ρ_1, ρ_2, &c., denote the distances of a variable point from given fixed points, l, m, &c. are constants, and k is a variable parameter, is cut orthogonally by the system

$$l\omega_1+m\omega_2+\text{\&c.} = \text{a constant},$$

where ω_1, ω_2, &c., are the angles which the lines joining the variable point to the fixed points make with a given direction. (Mr. M. Roberts.)

77. Show that the system of Cartesian ovals $l\rho_1+m\rho_2=c$, where l, m are constants, and c a variable parameter, is cut orthogonally by the curves

$$(\tan\tfrac{1}{2}\omega_1)^l = C(\tan\tfrac{1}{2}\omega_2)^m,$$

where ω_1, ω_2 are the base angles of the triangle formed by the variable point and the foci. (Mr. W. Roberts.)

78. Show that the envelopes of the lines

$$(x\cos\omega+y\sin\omega)\cos\omega+f(\omega)=c_1$$

are cut orthogonally by the envelopes of

$$(x\cos\omega+y\sin\omega)\cos\omega-f(\omega+\tfrac{1}{2}\pi)=c_2.$$

79. An arc of the lemniscate

$$r^2=a^2\cos 2\theta$$

is of given length; show that the locus of its centre of gravity is

$$(m+n)(x^2+y^2)^2=\frac{4a^4}{l^2}\{(m-1)x^2+(n-1)y^2\},$$

where l is the given length of the arc, and m, n are constants connected by the relation $m^2+n^2=2$.

80. Show that the system of curves

$$\left(r + \frac{k^2}{r}\right)^m (\cos\theta)^n = a,$$

where r, θ are polar co-ordinates, is cut orthogonally by the system

$$\left(r - \frac{k^2}{r}\right)^n (\sin\theta)^m = \beta.$$

81. Show that the differential equation

$$\frac{\mu\, d\mu}{(\mu^4 - c^2\mu^2)^{\frac{2}{3}}} + \frac{\nu\, d\nu}{(c^2\nu^2 - \nu^4)^{\frac{2}{3}}} = 0,$$

where μ, ν are elliptic co-ordinates, represents the evolutes of the system of confocal conics.

82. Show that

$$xy(a + b - 2c) + (c^2 - ab)(x + y) + 2abc - c^2(a + b) = 0,$$

and two similar relations are integrals of the differential equation

$$\frac{dx}{\{(x - a)(x - b)(x - c)\}^{\frac{2}{3}}} + \frac{dy}{\{(y - a)(y - b)(y - c)\}^{\frac{2}{3}}} = 0.$$

83. Given $u_1 + u_2 + u_3 + u_4 = K$, show that the anharmonic ratio of $\text{sn}^2 u_1$, &c., is equal to that of $\text{sn} u_1 \, \text{cn} u_1 / \text{dn} u_1$, &c., taken in the same order.

84. Given two curves, $\phi_n = 0$, $\phi_{n-2} = 0$, of the n^{th} and $(n - 2)^{th}$ degrees, respectively, show, as in Art. 168, that the sum of the areas between the curves $\phi_n = 0$, $\phi_n + k\phi_{n-2} = 0$, and two lines whose directions are θ_1, θ_2, is

$$k \int_{\theta_2}^{\theta_1} \frac{B_0}{A_0}\, d\theta,$$

where ϕ_n, ϕ_{n-2}, being transformed to polar co-ordinates, take the forms—

$$\phi_n \;\;= A_0 r^n \;\; + A_1 r^{n-1} + \&c.,$$

$$\phi_{n-2} = B_0 r^{n-2} + B_1 r^{n-3} + \&c.$$

85. If Σ is the algebraic sum of the areas intercepted between two lines, a curve of the n^{th} degree, and its n asymptotes; show that Σ can always be expressed by no higher transcendents than logarithms or circular functions.

86. U is a bicircular quartic and S is a circle. Show that the algebraic sum of the areas intercepted between two lines and the quartics $U=0$, $U+kS=0$ is proportional to the angle between the lines.

87. Show that the system of curves

$$(x^2+y^2)(x^2-y^2-k^2)=ax$$

is cut orthogonally by the system

$$y(y^2+3x^2-k^2)-\beta(x^2+y^2)=0.$$

88. If a curve be given by the equations

$$x \pm iy = \int \{c^2-(x' \pm iy')^2\}^n d(x' \pm iy'),$$

where x', y' lie on the Cassinian

$$(x'^2+y'^2)^2-2c^2(x'^2-y'^2)+k^4=0,$$

show that the arc of the locus is proportional to the corresponding arc of the Cassinian.

89. Show that the differential equation

$$\frac{du}{\sqrt{U}} \pm \frac{dv}{\sqrt{V}}=0,$$

where $\quad U=(au^2+2a'u+a'')(cu^2+2c'u+c'')-(bu^2+2b'u+b'')^2,$

$$V=(av^2+2bv+c)(a''v^2+2b''v+c'')-(a'v^2+2b'v+c')^2,$$

$$u=x+iy, \quad v=x-iy,$$

represents a system of bicircular quartics. Hence, also, show that two curves of the system pass through a point and cut each other orthogonally. (Darboux.)

90. Show that the differential equation

$$\frac{du}{\sqrt{\{(u-a)(u-b)(u-c)\}}} \pm \frac{dv}{\sqrt{\{(v-a)(v-b)(v-c)\}}}=0,$$

where $\quad x+iy=u, \quad x-iy=v,$

represent a system of Cartesian ovals.

91. Show that the differential equation

$$\frac{du}{\sqrt{(u^2 - c^2)}} \pm \frac{dv}{\sqrt{(v^2 - c^2)}} = 0,$$

where $$x + iy - u, \quad x - iy = v,$$

represent a system of confocal conics.

92. Show that the equations

$$\frac{du}{u^2 + c^2} \pm \frac{dv}{v^2 + c^2} = 0$$

represent two orthogonal systems of coaxal circles.

93. Show that the arc of the curve whose equation in polar co-ordinates is $r^4 = a^4 \cos 4\theta$ can be expressed in terms of two elliptic integrals of the first kind with numerical moduli.

94. Show that the curve determined by the equations

$$x + iy = \int \frac{(1 - z^2)^{2n+1}\, dz}{(1 + z^2)^{2n+2}}, \quad x - iy = i \int \frac{(1 + z^2)^{2n+1}\, dz}{(1 - z^2)^{2n+2}}$$

is algebraic, and that its arc is equal to an arc of the lemniscate. (Serret.)

95. Show that the system of curves

$$\frac{x^3}{\alpha} + \frac{xy^2}{\alpha - x} = c^2,$$

where α is a parameter, is cut orthogonally by the system

$$\frac{x^2 y}{y - \beta} - \frac{y^3}{\beta} = c^2.$$

96. Show that the envelope of the lines

$$x \cos \omega + y \sin \omega = \sqrt{\{\alpha + f(\omega)\}}$$

is cut orthogonally by the envelope of

$$x \cos \omega + y \sin \omega = \sqrt{\{\beta - f(\omega + \tfrac{1}{2}\pi)\}}.$$

97. Show that the differential equation

$$\frac{du}{\{(u-a)(u-b)(u-c)\}^{\frac{2}{3}}} + \frac{dv}{\{(v-a)(v-b)(v-c)\}^{\frac{2}{3}}} = 0,$$

where $$x + iy = u, \quad x - iy = v,$$

represents a system of curves of the sixth order. Show also that three curves of the system pass through a point, and that the tangents at the point are parallel to the sides of an equilateral triangle.

98. Show that the systems of curves

$$\frac{du}{(u^2 - c^2)^{\frac{2}{3}}} \pm \frac{dv}{(v^2 - c^2)^{\frac{2}{3}}} = 0,$$

where $x + iy = u$, $x - iy = v$, cut orthogonally. Show also that their equations in rectangular co-ordinates can be written, respectively,

$$(x^2 + y^2 + 2ax + 9c^2)^2 (x^2 + y^2 + 2ax) = 27 \{a (x^2 + y^2) + c^2 (2x + a)\}^2,$$

$$(x^2 + y^2 + 2\beta y - 9c^2)^2 (x^2 + y^2 + 2\beta y) = 27 \{\beta (x^2 + y^2) - c^2 (2y + \beta)\}^2.$$

99. Show that the area between any branch of the cubic

$$27xy \left(1 - \frac{x}{a} - \frac{y}{b}\right) = ab,$$

and the two asymptotes which include it, is one-third of the area of the triangle formed by the asymptotes.

100. Show that the arc of the curve whose polar equation is

$$r^6 - 2a^3 r^3 \cos 3\theta + k^6 = 0$$

can be expressed in terms of two elliptic integrals of the first kind.

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